Journal of Computational and Applied Mathematics 280 (2015) 265–274

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Asymptotic stability of Runge–Kutta methods for nonlinear differential equations with piecewise continuous arguments

X. Liu [a,*], M.Z. Liu [b]

[a] School of Mathematical Sciences, Liaocheng University, Liaocheng, 252059, China

[b] Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China

ARTICLE INFO

Article history:
Received 25 December 2013
Received in revised form 19 November 2014

Keywords:
Piecewise continuous arguments
Asymptotic stability
Runge–Kutta methods
Numerical solutions
Delay differential equations

ABSTRACT

This paper deals with the asymptotic stability of numerical solutions for differential equations with piecewise continuous arguments (EPCAs). The necessary and sufficient condition is given for non-confluent Runge–Kutta methods to preserve the stability of nonlinear scalar EPCAs. As for systems, we prove that some algebraically stable methods can preserve the asymptotic stability.

1. Introduction

This paper is concerned with the numerical stability of delay differential equations with piecewise continuous arguments (EPCAs)

$$x'(t) = f(t, x(t), x(\Phi(t))), \tag{1.1}$$

where $\Phi(t)$ is a piecewise continuous function. This equation arises in many biological models (see [1]). It is well known that this equation contains results on the relationship between equations with piecewise constant arguments and impulsive equations. The study of the properties of analytic solutions of EPCAs has been initiated in [2–5]. In the book of Wiener [6], the general theory and basic results for EPCAs have been listed. In recent years, some studies have been focused on the property of numerical solutions for EPCAs. In [7–11], the authors investigated the convergence and the stability of one-step methods for linear equations. In [12], the author improved the linear multistep methods so that the methods can preserve their convergent order for ODEs when applied to linear EPCAs. Recently, numerical stability of nonlinear EPCAs has attracted researchers' interest. In 2005, Li [13] discussed numerical asymptotical stability of a class of multistep methods. Wang studied the dissipativity and stability of Runge–Kutta methods for neutral differential equations in [14,15]. In this paper, we show that some high order methods, beside implicit Euler method and 2-Lobatto-IIIC method (see [15]), can preserve the stability of nonlinear EPCAs.

In this paper, we consider the equation:

$$\begin{cases} y'(t) = f(t, y(t), y([t])), & t \geq 0, \\ y(0) = y_0, \end{cases} \tag{1.2}$$

where $f : [0, +\infty) \times R^N \times R^N \to R^N$ is a continuous function and $[\cdot]$ denotes the greatest integer function.

* Corresponding author.
E-mail address: liuxing6213693@163.com (X. Liu).

http://dx.doi.org/10.1016/j.cam.2014.11.059

In [8], some properties of the solution of (1.2) are presented.

Definition 1 (*See [8]*). A solution of (1.2) on $[0, \infty)$ is a function $y(t)$ that satisfies the following conditions:

(i) $y(t)$ is continuous on $[0, \infty)$;
(ii) the derivative $y'(t)$ exists at each point $t \in [0, \infty)$, with the possible exception of the points $[t] \in [0, \infty)$ where one-sided derivatives exist;
(iii) (1.2) is satisfied on each interval $[n, n+1) \subset [0, \infty)$ with integral end-points.

Let $\langle \cdot, \cdot \rangle$ be an inner product on R^N and $\| \cdot \|$ the corresponding norm. We assume that there exist real constants α and β such that

$$\langle x_1 - x_2, f(t, x_1, y) - f(t, x_2, y) \rangle \le \alpha \|x_1 - x_2\|^2, \quad \forall t \ge 0,\ x_1, x_2, y \in R^N,$$

$$\|f(t, x, y_1) - f(t, x, y_2)\| \le \beta \|y_1 - y_2\|, \quad \forall t \ge 0,\ y_1, y_2, x \in R^N.$$

In order to study the stability of (1.2), we also consider the following equation

$$\begin{cases} z'(t) = f(t, z(t), z([t])), & t \ge 0, \\ z(0) = z_0. \end{cases} \tag{1.3}$$

Using the same technique as Theorem 2.2 in [8], we can obtain the following theorem:

Theorem 2. *If $\beta \le -\alpha$, then $\|y(t) - z(t)\| \le \|y_0 - z_0\|$ holds for all $t \ge 0$.*
If $\beta < -\alpha$, then $\lim_{t \to +\infty} \|y(t) - z(t)\| = 0$.

As usual, we expect the numerical solution to reproduce the property of the true solution.

Definition 3. A numerical method is called contractive for (1.2) if the numerical solutions y_n and z_n of (1.2) and (1.3) at the mesh points $t_n = nh,\ n \ge 0$ satisfies

$$\|y_n - z_n\| \le \|y_0 - z_0\|, \quad \forall n \in Z^+$$

for every stepsize h under the constraint $hm = 1$ where m is a positive integer.

Definition 4. A numerical method is called asymptotically stable for (1.2) if the numerical solutions y_n and z_n of (1.2) and (1.3) at the mesh points $t_n = nh,\ n \ge 0$ satisfies

$$\lim_{n \to +\infty} \|y_n - z_n\| = 0$$

for every stepsize h under the constraint $hm = 1$ where m is a positive integer.

2. Runge–Kutta methods

In this section we consider the adaption of the Runge–Kutta methods. Let $h = \frac{1}{m}$ be a given stepsize with integer $m \ge 1$ and the gridpoints t_n can be defined by $t_n = nh$ $(n = 0, 1, \ldots)$. Let (A, b, c) denote a given Runge–Kutta method with $\nu \times \nu$ matrix $A = (a_{ij})_{\nu \times \nu}$ and vectors $b = (b_1, \ldots, b_\nu)^T$, $c = (c_1, \ldots, c_\nu)^T$. The application of (A, b, c) in case of (1.2), yields

$$\begin{aligned} Y_i^n &= y_n + h \sum_{j=1}^{\nu} a_{ij} f(t_n + c_j h, Y_j^n, \overline{Y}_j^n), \quad i = 1, \ldots, \nu, \\ y_{n+1} &= y_n + h \sum_{i=1}^{\nu} b_i f(t_n + c_i h, Y_i^n, \overline{Y}_i^n), \end{aligned} \tag{2.1}$$

where $\overline{Y}_i^n$ is a given approximation to $y([t_n + c_i h])$. Denote $n = km + l$ $(l = 0, 1, \ldots, m-1)$. Then $\overline{Y}_i^n$ can be defined as y_{km} according to Definition 1 $(i = 1, \ldots, \nu)$. Therefore, (2.1) reduces to

$$\begin{aligned} Y_i^{km+l} &= y_{km+l} + h \sum_{j=1}^{\nu} a_{ij} f(t_{km+l} + c_j h, Y_j^{km+l}, y_{km}), \quad i = 1, \ldots, \nu, \\ y_{km+l+1} &= y_{km+l} + h \sum_{i=1}^{\nu} b_i f(t_{km+l} + c_i h, Y_i^{km+l}, y_{km}). \end{aligned} \tag{2.2}$$

Similarly application of the Runge–Kutta method in case of (1.3) yields

$$\begin{aligned} Z_i^{km+l} &= z_{km+l} + h \sum_{j=1}^{\nu} a_{ij} f(t_{km+l} + c_j h, Z_j^{km+l}, z_{km}), \quad i = 1, \ldots, \nu, \\ z_{km+l+1} &= z_{km+l} + h \sum_{i=1}^{\nu} b_i f(t_{km+l} + c_i h, Z_i^{km+l}, z_{km}). \end{aligned} \tag{2.3}$$