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Characterizing the finiteness of the Hausdorff distance between two algebraic curves[☆]

Angel Blasco, Sonia Pérez-Díaz [*]

Departamento de Física y Matemáticas, Universidad de Alcalá, E-28871 Madrid, Spain

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ABSTRACT

In this paper, we present a characterization for the Hausdorff distance between two given algebraic curves in the n-dimensional space (parametrically or implicitly defined) to be finite. The characterization is related with the asymptotic behavior of the two curves and it can be easily checked. More precisely, the Hausdorff distance between two curves $\mathcal{C}$ and $\overline{\mathcal{C}}$ is finite if and only if for each infinity branch of $\mathcal{C}$ there exists an infinity branch of $\overline{\mathcal{C}}$ such that the terms with positive exponent in the corresponding series are the same, and reciprocally.

1. Introduction

The Hausdorff distance is one of the most used measures in geometric pattern matching algorithms, computer aided design or computer graphics (see e.g. [1–4]).

Given a metric space (E, d) and two arbitrary subsets $A, B \subset E$, the Hausdorff distance assigns to each point of one set the distance to its closest point on the other and takes the maximum over all these values (see [5]). More precisely, the *Hausdorff distance* between A and B is defined as:

$$d_H(A, B) = \max\{\sup_{x\in A} \inf_{y\in B} d(x, y), \sup_{y\in B} \inf_{x\in A} d(x, y)\}.$$

In this paper, we deal with the particular case where $E = \mathbb{C}^n$ or $E = \mathbb{R}^n$, and d is the usual unitary or Euclidean distance (see Chapter 5 in [6]). In addition, the two arbitrary subsets are two real algebraic curves $\mathcal{C}$ and $\overline{\mathcal{C}}$. In this case, the Hausdorff distance between $\mathcal{C}$ and $\overline{\mathcal{C}}$ is given by

$$d_H(\mathcal{C}, \overline{\mathcal{C}}) = \max\{\sup_{p\in\mathcal{C}} d(p, \overline{\mathcal{C}}), \sup_{\overline{p}\in\overline{\mathcal{C}}} d(\overline{p}, \mathcal{C})\}, \quad \text{where } d(p, \mathcal{C}) = \min\{d(p, q) : q \in \mathcal{C}\}.$$

In general, $d_H(A, B)$ may be infinite, and some restrictions have to be imposed to guarantee its finiteness (see e.g. [7] or [8]).

As far as the authors know, there is no efficient algorithms for the exact computation of the Hausdorff distance between algebraic varieties (in fact, if both varieties are given in implicit form, the computation of the Hausdorff distance is even harder). Only some results for bounding or estimating the Hausdorff distance as well as computing it for some special cases

☆ The author S. Pérez-Díaz is member of the Research Group ASYNACS (Ref. CCEE2011/R34).

* Corresponding author.
E-mail addresses: angel.blasco@uah.es (A. Blasco), sonia.perez@uah.es (S. Pérez-Díaz).

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can be found. For instance, in [9], the Hausdorff distance between planar free-form curves using a polyline approach is estimated. More precisely, the input curves are approximated with polylines and the precise Hausdorff distance between polylines is computed. It is shown that the approximation error can be totally controlled. In [10], a method for computing the Hausdorff distance between two B-spline curves is developed. An estimation of the upper bound of the Hausdorff distance in an sub-interval is given, which is used to eliminate the sub-intervals whose upper bounds are smaller than the given lower bound. The conditions whether the Hausdorff distance occurs at an end point of the two curves are also provided, and these conditions are used to turn the Hausdorff distance computation problem between two curves into a minimum or maximum distance computation problem between a point and a curve, which can be solved as well. [11] defines and discusses the Hausdorff metric on the space of nonempty, closed, and bounded subsets of a given metric space. Two important topological properties are considered, completeness and boundedness. It is proved that each of these properties is possessed by a Hausdorff metric space if the property is possessed by the underlying metric space. The paper [12] is devoted to computational techniques for generating upper bounds on the Hausdorff distance between two planar curves (implicitly or parametrically defined). The bounds are computed directly from the control points (spline coefficients of the curves). Potential applications include error bounds for the approximate implicitization of spline curves, for the approximate parametrization of (piecewise) algebraic curves, and for algebraic curve fitting. This approach assumes that the two curves are fairly close to each other. In [2], a real-time algorithm for computing the precise Hausdorff distance between two planar freeform curves is presented. The algorithm is based on an effective technique that approximates each curve with a sequence of G^1 biarcs within an arbitrary error bound. In [8], authors consider real space algebraic curves, not necessarily bounded, whose Hausdorff distance is finite, and bounds of their distance are provided. These bounds are related to the distance between the projections of the space curves onto a plane. Thus, authors provide a theoretical result that reduces the estimation and bounding of the Hausdorff distance of algebraic curves from the spatial to the planar case.

In this paper, we deal with a different aspect concerning the Hausdorff distance. We do not deal with the computation or estimation of the Hausdorff distance as in the papers we mentioned above, but with a characterization on whether the Hausdorff distance between two given algebraic curves (parametrically or implicitly defined) in the n-dimensional space and that are not necessarily bounded, is finite. The characterization is based on the concept of *similar asymptotic behavior* introduced in this paper, and it improves Proposition 5.4 presented in [11] (see also [13]). The notion of similar asymptotic behavior has to deal with the convergence/divergence of the infinity branches of two given curves $\mathcal{C}$ and $\overline{\mathcal{C}}$. More precisely, we say that $\mathcal{C}$ and $\overline{\mathcal{C}}$ have a *similar asymptotic behavior* if there are no infinity branches in $\mathcal{C}$ which diverge from all the infinity branches in $\overline{\mathcal{C}}$, and reciprocally. In fact, we show that the Hausdorff distance between $\mathcal{C}$ and $\overline{\mathcal{C}}$ is finite if and only if both curves have *a similar asymptotic behavior*. This condition is very easy to formulate from the computational point of view and thus, we present an effective algorithm that checks if it holds.

Although the curves used in computer aided geometric design (CAGD) are usually bounded and there is no need to decide whether or not the Hausdorff distance is bounded, the characterization presented in this paper plays an important role in some other applications to CAGD as for instance in the approximate parametrization problem (see e.g. [7,14–17]). In that problem, given an affine curve $\mathcal{C}$ (say that it is a perturbation of a rational curve), the goal is to compute a rational proper parametrization of a rational affine curve $\overline{\mathcal{C}}$ near $\mathcal{C}$ (one may state the problem also for surfaces; see [18]). As one can check in the papers mentioned, the effectiveness of the algorithm depends on the closeness of $\mathcal{C}$ and $\overline{\mathcal{C}}$ and, at least, the finiteness of the Hausdorff distance between $\mathcal{C}$ and $\overline{\mathcal{C}}$ has to be guaranteed (which is equivalent to ensure a similar behavior of both curves at infinity). The potential applications of the results presented in this paper also include the approximate implicitization problem for curves and surfaces (see [19,20]).

Moreover, since this characterization is based on the notion of infinity branch which reflects the status of a curve at the points with sufficiently large coordinates, one may think in applying the results presented to the analysis of the behavior at infinity of an algebraic curve, which implies a wide applicability in many active research fields. For instance, the following problems could be considered: sketch the graph of a given algebraic curve as well as to study its topology (see e.g. [21–24]), compute the shapes in a family of space curves (see [25]), determine the symmetries of a given curve (see [26]), etc.

The structure of the paper is as follows: in Section 2, we present the terminology that will be used throughout the paper as well as some previous results. In this section, we assume that the given curve is implicitly defined but one can easily check that the results obtained are independent on the representation of the curve. Only computational aspects change, and thus, in Section 3, we present the necessary computational techniques to deal with curves parametrically defined. Section 4 is devoted to present the main theorem where the finiteness of the Hausdorff distance is characterized. For this purpose, some technical lemmas are proved. In Section 5, we derive an algorithm that decides whether the Hausdorff distance between two given algebraic curves is finite and we show how this algorithm can be adapted to be applied only to the real parts of the given curves. We illustrate the method with some examples in detail. Moreover, some practical examples are also shown where one can check the applicability of our results to problems in CAGD. We finish with a section of conclusions (Section 6) where we summarize the results obtained, we emphasize the new contributions of this paper, and we propose topics for further study.

2. Previous results for implicit space curves

In this section, we present some previous definitions and results concerning curves in the n-dimensional space. We assume that the curves are defined by a finite set of real polynomials but we will see that all the results and concepts