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An energy-conserving second order numerical scheme for nonlinear hyperbolic equation with an exponential nonlinear term

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ABSTRACT

We present a second order accurate numerical scheme for a nonlinear hyperbolic equation with an exponential nonlinear term. The solution to such an equation is proven to have a conservative nonlinear energy. Due to the special nature of the nonlinear term, the positivity is proven to be preserved under a periodic boundary condition for the solution. For the numerical scheme, a highly nonlinear fractional term is involved, for the theoretical justification of the energy stability. We propose a linear iteration algorithm to solve this nonlinear numerical scheme. A theoretical analysis shows a contraction mapping property of such a linear iteration under a trivial constraint for the time step. We also provide a detailed convergence analysis for the second order scheme, in the $\ell^\infty(0,T;\ell^\infty)$ norm. Such an error estimate in the maximum norm can be obtained by performing a higher order consistency analysis using asymptotic expansions for the numerical solution. As a result, instead of the standard comparison between the exact and numerical solutions, an error estimate between the numerical solution and the constructed approximate solution yields an $O(\Delta t^3 + h^4)$ convergence in $\ell^\infty(0,T;\ell^2)$ norm, which leads to the necessary ℓ^∞ error estimate using the inverse inequality, under a standard constraint $\Delta t \le Ch$. A numerical accuracy check is given and some numerical simulation results are also presented.

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1. Introduction

Consider the following nonlinear hyperbolic equation in the domain $\Omega = [0, L]^2$, with a periodic boundary condition,

$$\begin{cases} \partial_t^2 u - \Delta u = \alpha e^{-u}, & \text{in } \Omega_T, \\ u\,|_{t=0} = C_0, & \partial_t u\,|_{t=0} = 0. \end{cases} \tag{1.1}$$

Here Ω_T is defined as $\Omega \times (0, T]$, $\alpha = \alpha(x, y)$ is a periodic non-negative function with finite upper bound $\bar{\alpha}$ and is differentiable with respect to space up to the rth order, and C_0 is a non-negative constant. An integral equation arises from Johnson–Mehl–Avrami–Kolmogorov theory [1–3] and Cahn's time cone method [4], which characterizes the nucleation and growth phenomenon of nuclei. In the framework of the time cone method, u stands for the number of nuclei and satisfies an integral equation in the time–space region. Recently, Liu and Yamamoto successfully proved that the integral equation

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could be reduced to a hyperbolic system [5], which is exactly the same as (1.1) in one dimensional case. In addition, similar nonlinear hyperbolic systems can be derived in high dimensional cases. The righthand term αe^{-u} describes the nucleation rate. The introduction of the nonlinear term e^{-u} is an available assumption based on the physical fact that nucleation rate decreases with the increase of the number of nuclei.

Similar to the linear problem, the nonlinear equation (1.1) also admits an energy law. Indeed, we can define the nonlinear energy as follows,

$$E(t) = \int_{\Omega} \left(\frac{1}{2} |\partial_t u|^2 + \frac{1}{2} |\nabla u|^2 + \alpha e^{-u} \right) d\mathbf{x} = \frac{1}{2} \left(\|\partial_t u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \right) + \int_{\Omega} \alpha e^{-u} d\mathbf{x}. \tag{1.2}$$

By taking inner product with $\partial_t u$ for the equation of (1.1), it is straightforward to observe that

$$E'(t) \equiv 0, \quad \text{i.e.}\ E(t) = E(0),\ \forall t > 0, \tag{1.3}$$

which shows a conservation for the energy defined as (1.2).

Although the energy $E(t)$ is always conserved for all time, we should point out that the existence and uniqueness of the solution of (1.1) are not straightforward. Similar energy conservation also holds if the exponential nonlinear term e^{-u} is replaced by a polynomial one $|u|^p$, while the solution may not exist globally in time and blow up once p exceeds a critical value [6]. Up to now, the existence of a global (in time) solution of (1.1) with a general boundary condition and initial value has been an open problem. In this paper we prove there exists a global in time, unique positive solution of (1.1) with periodic boundary conditions and special initial values. The positivity property is important to prove the well-posedness of the problem, and a new metric of the solution space is introduced to establish the global (in time) existence. On the other hand, the positivity property is also required by the physical problem, since the solution corresponds to the expected number of nuclei, which is expected to be positive. The technique used here cannot be directly applied to the equation with Dirichlet boundary condition, since the solution may not always be positive and the well-posedness is not clear in two or three dimensional cases. Meanwhile, the local existence can be established in enough small time interval, and the numerical scheme also works well for the equation with Dirichlet boundary condition.

There are many numerical schemes to solve nonlinear hyperbolic equations, for example, see [7–9]. However, very few have dealt with an exponential nonlinear term. Due to the special nature of the nonlinear term appearing in (1.1), the positivity property is proven to be preserved under a periodic boundary condition. Based on the positivity, we present an energy-conserving second order numerical scheme. For this scheme, a highly nonlinear fractional term is involved, for the theoretical justification of the energy stability. For the implementation part, we propose a linear iteration method to solve this nonlinear numerical scheme. A theoretical analysis will be given to prove a contraction mapping property of such a linear iteration under a trivial constraint for the time step.

We also provide a detailed convergence analysis for the second order scheme, in the $\ell^\infty(0, T; \ell^\infty)$ norm. Such an error estimate in the maximum norm can be obtained by performing a higher order consistency analysis using asymptotic expansions for the numerical solution. As a result, instead of the standard comparison between the exact and numerical solutions, an error estimate between the numerical solution and the constructed approximate solution yields an $O(\Delta t^3 + h^4)$ convergence in $\ell^\infty(0, T; \ell^2)$ norm, which leads to the necessary ℓ^∞ error estimate using the inverse inequality, under a standard constraint $\Delta t \le Ch$, where C stands for some given constant.

The rest of the paper is organized as follows. In Section 2, we discuss the well-posedness and positivity of the solution to problem (1.1). In Section 3, the second order scheme is proposed and the energy-conserving property is proven. In Section 4, to facilitate the numerical implementation, we modify the numerical scheme based on the positivity of the solution and discuss some details in the implementation process. Next we give the corresponding convergence analysis in Section 5. Subsequently, several numerical examples are carried out in Section 6. Finally, some conclusions are made in Section 7.

2. Well-posedness and positivity

In this section, we prove the well-posedness and positivity of the solution to (1.1). The main tool we use here is Banach's fixed point theorem [10]; also see the related discussions in [11].

Let $C(0, T; C_\#(\Omega))$ denote the space of all functions which are continuous with respect to both time and space. Herein and what after, the subscript # means that the functions in the corresponding space are periodic with respect to space, i.e. for any function f, there exists its periodic extension defined in $\mathbb{R}^2$, still denoted as f, which satisfies

$$f(x + mL, y + nL) = f(x, y), \tag{2.1}$$

with any integers m and n. We denote X as the subspace of $C(0, T; C_\#(\Omega))$, which contains all non-negative functions.

Besides, we introduce the following metric. For any functions $f, g \in C(0, T; C_\#(\Omega))$, define

$$\delta(f, g) = \max_{t \in [0,T]} \left(e^{-st} \|f(x, y, t) - g(x, y, t)\|_{C(\Omega)} \right), \tag{2.2}$$

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