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Error analysis of waveform relaxation method for semi-linear partial differential equations

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ABSTRACT

The waveform relaxation (WR) method is investigated for semi-linear partial differential equations. Explicit error estimation is given for the iteration error. A way to combine WR with convergent numerical methods is proposed, the error of the combined method is analyzed and its convergence is proven. The effect of the application of time windows is discussed. Numerical tests are presented to confirm the theoretical results.

1. Introduction

The method of waveform relaxation (WR) is an iterative method which can be applied for a large class of problems. The first description of the method can be found in [1], where it was used to approximate the solution of a system of ordinary differential equations (ODEs) describing large scale circuits. Since then many works have been devoted to investigate the convergence of the method for different types of problems, for example [2–4]. All of them consider time dependent problems that are either a system of ODEs originally or obtained from partial differential equations (PDEs) by spatial discretization. The key to prove convergence in every case is the Lipschitz property of the function acting on the right hand side. A different approach, the Schwarz waveform relaxation algorithm has been proposed to solve partial differential equations in [5–7].

In PDEs describing diffusion or advection processes the spatial differentiation is not a Lipschitz-continuous operation. Consequently the usual formulation of WR fails when the method is applied directly on reaction–diffusion or reaction–advection problems. Furthermore the convergence rates of the spatially discretized problem depend on the discretization parameter thus the results cannot automatically be transferred to the original continuous model. The deterioration of convergence rates was highlighted in [8], where WR was used directly to a reaction–diffusion equation in one spatial dimension. The proposed iteration in [8] provides faster convergence rate valid for PDEs. By providing better error estimates than the existing ones for the traditional WR process, [5] suggests that faster convergence can be achieved with a structurally different approach, namely the decomposition of the space domain, not the acting operator in the equation.

The present work is an extension of [8] in two respects: using the concept of strongly continuous one-parameter semigroups, a large class of problems is discussed including systems of reaction–diffusion and reaction–advection equations in multiple spatial dimensions; the given error estimations are refined and explicit—meaning that they do not contain the solution of the problem itself.

The proposed application of the WR method here is to define the iteration without discretization, then solving the subproblem of each iteration numerically, that is considering discretization of the subproblems. This scenario allows to

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investigate the effect of the numerical treatment as well, an overall error estimation can be formulated which includes the iteration error and the *cumulative numerical error*.

The estimation of the iteration error suggests that faster convergence can be achieved by dividing the time interval into subintervals and applying WR on these time windows one after another. This procedure is called *windowing*. Result on convergence of windowing for ODEs is given in [9], the windowing technique is employed to accelerate the convergence of the WR process in [10]. Here as an extension of these results, the convergence of windowing is proven for a large class of PDEs.

After defining the class of problems that is the subject of this study Section 2 provides improved error estimates for the applied WR which is the same as in [8]. In Section 3 the cumulative numerical error is introduced and an estimation of the overall error is formulated. In the last parts of Sections 2 and 3 the results are formulated for reaction–diffusion problems specifically. Section 4 contains the results on the windowing process. Numerical test results are given in Section 5 to illustrate the theoretical results of the previous Sections.

2. Waveform relaxation for semi-linear PDEs

Let $(X, \|\cdot\|)$ be a Banach space, $D(A), D(F) \subset X$, suppose that $\Omega := D(A) \cap D(F)$ is an open connected set, $u_0 \in \Omega$, $T \in \mathbb{R}^+$. Let $A : D(A) \longrightarrow X$ be linear and $F : D(F) \longrightarrow X$ a nonlinear operator. Consider the initial value problem

$$u'(t) = Au(t) + F(u(t)), \quad u(0) = u_0, \tag{1}$$

with $t \in [0, T]$. The derivative at $t = 0$ should be understood as the right hand derivative.

A *classical solution* of (1) is a continuously differentiable function u for which $u(t) \in \Omega$ for all $t \in [0, T]$. This solution is also the solution of the integral equation

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))\mathrm{d}s \quad \text{for every } t \in [0, T] \tag{2}$$

with a properly defined set of bounded linear operators $\{S(t) : X \to X, t \in [0, T]\}$. Let the waveform relaxation method for (1) be defined as

$$v_i'(t) = Av_i(t) + F(v_{i-1}(t)), \quad v_i(0) = u_0, \ t \in [0, T], \tag{3}$$

where $i \in I := \{1, 2, \ldots, m\}$, with some $m \in \mathbb{N}$ (the number of iterations) and the starting iteration function $v_0(t) = u_0$ for every $t \in [0, T]$. With the same operators $\{S(t)\}_{t\in\mathbb{R}_0^+}$ the solutions of the subproblems of (3) can be expressed as

$$v_i(t) = S(t)u_0 + \int_0^t S(t-s)F(v_{i-1}(s))\mathrm{d}s. \tag{4}$$

The following two *assumptions* form the framework of this study, they will remain valid throughout this paper. Suppose that

1. The operator A generates a strongly continuous semigroup $S(t)$, with $\|S(t)x\| \leqslant Me^{\omega t}\|x\|$, for every $x \in X$ and $t \in [0, T]$, where M and ω are nonnegative constants.
2. There is a closed ball $B_\delta(u_0)$ with $\delta \in \mathbb{R}^+$ and there is a constant L such that $\|F(v) - F(w)\| \leqslant L\|v - w\|$ for every $v, w \in B_\delta(u_0)$.

Under these assumptions (2) has a unique solution u, such that there is $t_\delta \in \mathbb{R}^+$ for which $\{u(t), t \in [0, t_\delta]\} \subset B_\delta(u_0)$. It is easy to see that if there is a classical solution to (1), then it is the solution of (2) as well. The existence of the solution of (2) is investigated in many works such as [11] or [12].

For an extensive description of theory of semigroups see [13]. Here let us just recall a basic relation that is often used in this section, see for example [13, p. 50]:

Lemma 1. *If A generates a strongly continuous semigroup $\{S(t)\}_{t\in\mathbb{R}_0^+}$ of bounded linear operators so that with $\|S(t)x\| \leqslant Me^{\omega t}\|x\|$, for every $x \in X$ and $t \in \mathbb{R}_0^+$, then for every $x \in D(A)$*

$$S(t)x - x = \int_0^t S(s)Ax\mathrm{d}s.$$

2.1. Iteration error

Using the concept of operator semigroup the results of [8] are extended to a large class of problems like (1). Furthermore by deriving an estimate for $\|u(t) - u_0\|$ a sharper and explicit upper bound is provided for the iteration error, as opposed to the estimate containing $\sup_{t\in[0,T]} \|u(t) - u_0\|$ used in the literature.