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Computation of integrals with oscillatory singular factors of algebraic and logarithmic type[☆]

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ABSTRACT

In this paper, we present the Clenshaw–Curtis–Filon methods and the higher order methods for computing many classes of oscillatory integrals with algebraic or logarithmic singularities at the two endpoints of the interval of integration. The methods first require an interpolant of the nonoscillatory and nonsingular parts of the integrands in $N + 1$ Clenshaw–Curtis points. Then, the required modified moments, can be accurately and efficiently computed by constructing some recurrence relations. Moreover, for these quadrature rules, their absolute errors in inverse powers of the frequency ω, are given. The presented methods share the advantageous property that the accuracy improves greatly, for fixed N, as ω increases. Numerical examples show the accuracy and efficiency of the proposed methods.

1. Introduction

In this paper we are concerned with the numerical evaluation of oscillatory integrals of the forms

$$I_1[f] = \int_0^1 f(x)G^{[c]}(x)e^{i\omega x}dx, \tag{1.1}$$

$$I_2[f] = \int_0^1 f(x)G^{[c]}(x)J_m(\omega x)dx, \tag{1.2}$$

$$I_3[f] = \int_0^1 f(x)G^{[c]}(x)Ai(-\omega x)dx, \tag{1.3}$$

where f is a sufficiently smooth function on $[0, 1]$, ω is a positive parameter, $G^{[c]}(x)$ is a product of singular factors of algebraic or logarithmic type at the two endpoints 0 and 1, $J_m(z)$ is the Bessel function of the first kind and of order m with $Re(m) > -1$, and $Ai(x)$ is an Airy function. Here, weight functions $G^{[c]}(x)$ and parameters are listed in Table 1. The above integrals arise widely in mathematical and numerical modeling of oscillatory phenomena in many areas of sciences and engineering such as astronomy, electromagnetics, acoustics, scattering problems, physical optics, electrodynamics, computerized tomography, and applied mathematics [1–10].

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Table 1
Weight functions $G^{[c]}(x)$ and parameters.

Number c	$G^{[c]}(x)$	Parameters
1	$\ln(x)x^{\alpha}(1-x)^{\beta}$	$\alpha, \beta > -1$
2	$x^{\alpha}(1-x)^{\beta}\ln(1-x)$	$\alpha, \beta > -1$
3	$\ln(x)x^{\alpha}(1-x)^{\beta}\ln(1-x)$	$\alpha, \beta > -1$
4	$(\ln(x)^{s})^{p}x^{\alpha}(1-x)^{\beta}(\ln(1-x)^{t})^{q}$	$\alpha, \beta > -1, s, t \in \mathcal{R}, p, q = 0, 1, 2, \ldots$

The integrals of the forms 1–3 in Table 1 belong to the special cases of the form 4. The integrals 1–4 with the weak singularities arise in the numerical approximations of solutions to Volterra integral equations of the first kind involving highly oscillatory kernels with weak singularities [3,4]. In addition, it is well-known that the Radon transform, which plays an important role in the CT, PET and SPECT technology of medical sciences and is widely applicable to tomography, the creation of an image from the scattering data associated to cross-sectional scans of an object, is closely related to this form of oscillatory singular integrals [9,10]. The singularities in (1.1)–(1.3) are also called singularities of the Radon transform in medical tomography. Moreover, the integrals 1–4 are also used to the solution of the singular integral equation for classical crack problems in plane and antiplane elasticity [8]. Further, they can be taken as model integrals appearing in boundary integral equations for high-frequency acoustic scattering (e.g., high-frequency Helmholtz equation in two dimensions), where the kernels have algebraic or logarithmic singularities on the diagonal ([6,7] and references therein), which is also our main target application.

The singularities of algebraic and logarithmic type, and possible high oscillations of the integrands make the integrals (1.1)–(1.3) very difficult to approximate accurately using standard methods, e.g., Gaussian quadrature rule. It should be also pointed out that many efficient methods, such as the *Filon-type* [11,12], *Levin-type* [13] and *numerical steepest descent methods* [14], cannot be applied directly to the integrals of the forms 1–4 in Table 1, since the nonoscillatory parts of the integrands are undefined at the two endpoints of the interval. Fortunately, in this paper we can introduce the *Clenshaw–Curtis–Filon methods* and the higher order methods for computing (1.1)–(1.3). The methods presented in this paper first require an interpolant of $f(x)$ in the Clenshaw–Curtis points, based on the ideas presented in [15]. Then, for singular weight functions $G^{[c]}(x)$ of the forms 1–4 in Table 1, the required modified moments in the proposed methods,

$$\int_0^1 G^{[c]}(x)T_j^*(x)S(\omega, x)dx, \tag{1.4}$$

can be accurately and efficiently computed by constructing some recurrence relations, where $S(\omega, x) = e^{i\omega x}, J_m(\omega x), Ai(-\omega x)$, respectively, and $T_j^*(x)$ denotes the shifted Chebyshev polynomial of the first kind of degree j on $[0, 1]$. Consequently, the presented methods allow us to overcome some problems that usually appear in the case when the integrands are both singular and oscillating.

Here, we would also like to mention several other papers related to the integrals considered in this article. On one side, some singular Fourier integrals are well-studied in the following literatures. Firstly, a fast algorithm for computing the integral $\int_a^b (x-a)^{\alpha}(b-x)^{\beta}f(x)e^{i\omega x}dx$, $\alpha, \beta > -1$ is proposed in [16]. Then, the *Filon-type, Clenshaw–Curtis–Filon-type, Chebyshev expansions, numerical steepest descent methods* for computing Fourier integrals $\int_a^b (x-a)^{\alpha}(b-x)^{\beta}f(x)e^{i\omega x}dx$, $\alpha, \beta > -1$, are also given in [17–19]. The recent works [20,21] present the Filon–Clenshaw–Curtis rules for the integrals $\int_a^b f(x)e^{i\omega x}dx$, and $\int_{-1}^1 f(x)\ln[(x-a)^2]e^{i\omega g(x)}dx$, $-1 \le a \le 1$, where f may have algebraic singularities and g has a stationary point on the interval of integration. Here, the use of graded meshes toward the singularities has shown to be also efficient in [20,21]. The authors in [22] present the *Clenshaw–Curtis–Filon methods* for approximating two classes of the Fourier integrals $\int_0^b x^{\alpha}f(x)e^{i\omega x}dx$ and $\int_0^b x^{\alpha}\ln(x)f(x)e^{i\omega x}dx$, $\alpha > -1$ with a singularity at one endpoint 0. On the other hand, in recent years, there has been tremendous interest in developing numerical methods for the nonsingular or singular Bessel integrals. The authors in [23,24], design the *Clenshaw–Curtis–Filon-type method* based on Clenshaw–Curtis points, for computing Bessel transforms $\int_a^b f(x)J_m(\omega x)dx$ without singularity. Moreover, the orthogonal expansion method [25], the *Filon-type method* [26], and the *Clenshaw–Curtis–Filon method* [26,27] are given for approximating the Bessel transform $\int_0^1 x^{\alpha}(1-x)^{\beta}f(x)J_m(\omega x)dx$, $\alpha, \beta > -1$, with singularities at the two endpoints, respectively. Recently, in [28], the authors of this paper have presented a *Clenshaw–Curtis–Filon method* and a *Clenshaw–Curtis–Filon-type method* for the integrals $\int_0^b x^{\alpha}f(x)J_m(\omega x)dx$ and $\int_0^b x^{\alpha}\ln(x)f(x)J_m(\omega x)dx$, $\alpha > -1$, with algebraic or logarithmic singularities at one endpoint 0, and, more importantly, for these quadrature rules the authors derive new computational sharp error bounds by rigorous proof. And, Xu and Xiang in [29] propose the *Clenshaw–Curtis–Filon-type method* for computing the oscillatory Airy integrals $\int_0^1 x^{\alpha}(1-x)^{\beta}f(x)Ai(-\omega x)dx$, $\alpha, \beta > -1$, with singularities at the two endpoints. Finally, it should be also noted that the three recurrence relations presented in [16,27,29] play a key role in the construction of the devised methods in this paper.

The outline of this paper is as follows. In Section 2 we present the Clenshaw–Curtis–Filon method for computing the singular Fourier transforms (1.1). In Section 3, the proposed method is extended to compute the singular oscillatory Bessel transforms (1.2) and Airy transforms (1.3). Section 4 gives higher order methods and their error analysis in inverse powers