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Implementation of Neumann boundary condition with influence matrix method for viscous annular flow using pseudospectral collocation

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ABSTRACT

The flowfield of an annular Couette flow is predicted numerically from an unsteady initial condition using the Chebyshev–Fourier collocation method. The numerical solution is obtained from the vorticity–velocity formulation of the unsteady Navier–Stokes equations. In this formulation the velocity boundary conditions are overspecified while vorticity boundary conditions are unspecified. This difficulty is resolved by a matrix influence method to convert the overspecified velocity boundary conditions to sufficiently specified Dirichlet boundary conditions for both velocity and vorticity. These boundary conditions are implemented by considering three methods: the traditional row replacement method, Fornberg's fictitious points method, and Driscoll and Hale's rectangular collocation method. The solution is advanced in time using the third-order Adams–Bashforth semi-implicit backward differentiation scheme. The accuracy of the numerical solutions are assessed in two ways. In one case, the accuracy of the converged steady state solution is compared to the analytical solution. In the second case, the residual of the continuity equation for the vorticity–velocity method is assessed in comparison to the vorticity streamfunction formulation that inherently satisfies the continuity equation. All methods demonstrate high accuracy for the continuity equation residual at the domain interior; however, the row replacement and fictitious points methods exhibit poor accuracy at the boundaries during the transient. The rectangular projection method exhibits excellent accuracy throughout the domain and boundaries at all times using the resampling points. On the other hand, for evaluating secondary quantities such as the wall shear stress, the rectangular projection method demonstrates several orders of magnitude less accuracy when evaluated using the resampling points, but high accuracy when evaluated directly at the original collocation points.

1. Introduction

The problem of annular Couette flow is studied as a vehicle for developing pseudospectral collocation methods to solve the unsteady Navier–Stokes equations in the vorticity–streamfunction and vorticity–velocity formulations for the radial and

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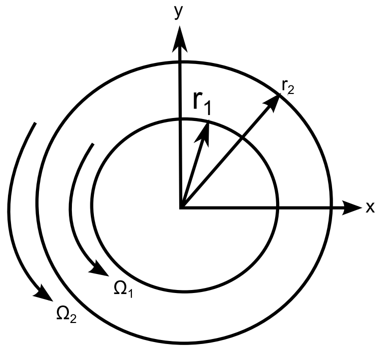

Fig. 1. Annular Couette flow in Cartesian coordinates.

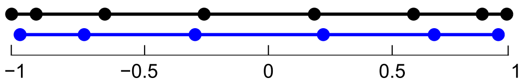

Fig. 2. Typical Chebyshev spacing for rectangular collocation method; top line represents a grid of N Chebyshev points of the second kind, the bottom line represents the $N - N_{BC}$ Chebyshev points of the first kind.

azimuthal velocities in time and space subject to proper boundary conditions. Notwithstanding its engineering importance, the solution in time and space is relatively scarce, but exact solutions exist under simplified cases. For example, the exact solution for the spatial azimuthal velocity is already established under steady state conditions [1]. On the other hand, the unsteady formulation for the azimuthal velocity as a function of the radial coordinate and time is suggested in terms of a partial differential equation governing vorticity, which has the same form as the classical heat conduction equation [2]. The problem formulation in time and space, radially and azimuthally, is attractive numerically for a number of reasons. First, the steady-state annular Couette flow analytical solution exists, which provides a reliable basis for assessing the accuracy and convergence of the developed solution through transients. Second, the domain is geometrically similar to that of flow around a cylinder, assuming a circular farfield boundary, which is of great interest as a basis for steady solutions [3], and unsteady solutions of vortex shedding [4]. Third, the pseudospectral method for a Couette flow between concentric cylinders is extendable to eccentric cylinders [5]. Last but not least, the impervious domain, which limits the boundary conditions (BCs) to no-slip walls, can be extended to a Couette flow with porous BCs [6].

Insofar as the pseudospectral method for the solution of the Navier–Stokes flow problem is concerned, much attention was on nonprimitive variables, namely the vorticity–streamfunction formulation, referred to herein as the $\omega - \psi$ formulation [7], as a system of two partial differential equations (PDE). The traditional challenge with a $\omega - \psi$ formulation is with regard to the lack of BCs for ω at no-slip walls, while the BCs for $\omega - \psi$ are overdefined, so much so that imposing improper or inconsistent BCs can lead to unstable solutions [8]. Furthermore, BCs inconsistencies that are mitigated in periodic problems can lead to instabilities in nonperiodic problems [2]. In that regard, Napolitano, et al., [9] provide an excellent review of BCs for the $\omega - \psi$ formulation despite the treatment limited to the finite difference and finite element methods. On the other hand, Boyd and Flyer [10] offer a nice exposition on compatible initial and boundary conditions for 2D incompressible periodic flow in a channel based on the Green's function. The restriction of the $\omega - \psi$ formulation to 2D is resolved by resorting to a general hybrid primitive and nonprimitive variables in 3D, namely the vorticity–velocity formulation, referred to herein as the $\omega - v$ formulation in which the system of PDEs over the domain of interest consists of the curl of velocity, vorticity equation and continuity equation. Obviously, the latter equation is not satisfied directly as in the $\omega - \psi$ formulation. In spite of the advantage of the $\omega - v$ formulation in 3D, the imposition of a set of compatible initial and boundary conditions is necessary. Quartapelle [11] provides a thorough coverage of compatibility conditions, as well as alternative formulations with the necessary BCs, primarily for open, bounded and simply connected fluid domain. For instance, one of the formulations in 2D consists of two PDEs, the vorticity equation, and the Laplacian of velocity in lieu of continuity equation. Various schemes to solve the set of PDEs include the second-order centered difference method [12] along with the influence matrix method for the BCs [13]; spectral-hp method [14], and the kinematic Laplacian equation method, which decouples vorticity in time from velocity in a finite element space [15] to name a few methods. Other methods to decouple pressure from velocity in the Navier–Stokes equations include the projection method [16] and the gauge method [17,18], which are outside the scope of this paper. Other methods, such as the fourth-order streamfunction formulation [19], are also outside the scope of this paper.

In this paper, the rectangular collocation method [20] is applied to the $\omega - v$ formulation of an unsteady Couette flow in an annular domain defined by $\Omega = [r_1, r_2] \times [0, 2\pi]$, as shown in Fig. 1. The set of PDEs consists of the vorticity equation, curl of velocity and continuity equation. In this method, the computation process, coined resampling of a polynomial interpolant by Driscoll and Hale [20], is carried out by projecting the differential and other operators at $N - N_{BC}$ points, as shown in Fig. 2, where N is the number of the original Chebyshev grid points and N_{BC} is the number of BCs.