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Asymptotics for the random coefficient first-order autoregressive model with possibly heavy-tailed innovations

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ABSTRACT

Consider a random coefficient AR(1) model, $X_t = (\rho_n + \phi_n)X_{t-1} + u_t$, where $\{\rho_n, n \geq 1\}$ is a sequence of real numbers, $\{\phi_n, n \geq 1\}$ is a sequence of random variables, and the innovations of the model form a sequence of i.i.d. random variables belonging to the domain of attraction of the normal law. By imposing some weaker conditions, the conditional least squares estimator of the autoregressive coefficient ρ_n is achieved, and shown to be asymptotically normal by allowing the second moment of the innovation to be possibly infinite.

1. Introduction

Let $\{X_t, t \geq 1\}$ be a first-order autoregressive (AR(1)) time series defined recursively by

$$X_t = \rho X_{t-1} + u_t, \quad t = 1, 2, \ldots, n, \tag{1.1}$$

where X_t is the observation at time t, ρ is an unknown parameter and $\{u_t, t \geq 1\}$ is a sequence of independent and identically distributed (i.i.d.) random variables. Given the observations $X_0, X_1, \ldots, X_n$, ρ is customarily estimated by its least squares estimator (LSE):

$$\hat{\rho} = \frac{\sum_{t=1}^{n} X_{t-1}X_t}{\sum_{t=1}^{n} X_{t-1}^2}. \tag{1.2}$$

As we know, if $\mathsf{E}u_1 = 0$ and $\mathsf{E}u_1^2 < \infty$, then the AR(1) process (1.1) with $|\rho| < 1$ is asymptotically stationary and the LSE $\hat{\rho}$ satisfies that

$$\sqrt{n}(\hat{\rho} - \rho) \xrightarrow{d} N(0, 1 - \rho^2), \quad \text{as } n \to \infty \tag{1.3}$$

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(see [1]), where $\xrightarrow{d}$ ($\xrightarrow{p}$) denotes convergence in distribution (in probability). When $\rho = 1$, model (1.1) becomes a unit root process, which is non-stationary. Since the unit root phenomenon widely exists in economics, many statisticians have investigated the asymptotic properties of the $\hat{\rho}$ in this case, and it is shown that

$$n(\hat{\rho}-1) \xrightarrow{d} \frac{W^2(1)-1}{2\int_0^1 W^2(t)\mathrm{d}t}, \quad \text{as } n \to \infty,$$

where $\{W(t), t \geq 0\}$ is a standard Brownian motion (see [2,3]).

Lately, motivated by bridging the $\sqrt{n}$ and n convergence rates for the stationary and non-stationary cases, Phillips and Magdalinos [4] proposed an AR(1) model similar to model (1.1) by taking $\rho = \rho_n := 1 + c/k_n$ with $c < 0$, where $\{k_n, n \geq 1\}$ is a sequence increasing to ∞ such that $k_n = o(n)$ as $n \to \infty$. For $c < 0$, Phillips and Magdalinos [4] obtained that

$$\sqrt{nk_n}(\hat{\rho}-\rho) \xrightarrow{d} N(0, -2c), \quad \text{as } n \to \infty, \tag{1.4}$$

under the i.i.d. assumption of $\{u_t, t \geq 1\}$ with $\mathrm{E}u_1 = 0$ and $\mathrm{E}|u_1|^{2+\delta} < \infty$ for some $\delta > 0$.

Also we note that during the past decades, there has been an increasing interest in nonlinear time series models. One of the examples is the random coefficient model, which was introduced and studied by Nicholls and Quinn [5]. Random coefficient autoregressive (RCA) time series have been invented in the context of random perturbations of dynamical systems, but are now used in a variety of applications; for example, in finance and biology (see [6]).

Obviously, RCA time series are generalizations of autoregressive time series, since they allow for randomly disturbed coefficients as well, and hence many researchers devoted themselves to the studying of asymptotic properties of RCA time series. For the early study of RCA time series, the reader is referred to Anděl [7], Conlisk [8], Robinson [9], Nicholls and Quinn [10,11], Feigin and Tweedie [12], Weiss [13], Guyton et al. [14], Kim and Basawa [15], Hwang and Basawa [16,17], Lee [18], among others. Recently, Leipus and Surgailis [19] studied the long-memory properties and the partial sums process of the RCA time series, Distaso [20] proposed new tests for simple unit root and unit root with a possibly non-zero drift process in RCA models, Thorsten and Jens-Peter [21] investigated the bootstrap of RCA models, while Zhang and Yang [22] and Zhao et al. [23] studied the conditional least squares estimator of a generalized RCA model, respectively.

In fact, Zhang and Yang [22] introduced the following generalized random coefficient first-order autoregressive (RCA(1)) time series, which can been viewed as a natural generalization of Phillips and Magdalinos [4],

$$X_t = (\rho_n + \phi_n)X_{t-1} + u_t, \quad t = 1, 2, \ldots, n, \tag{1.5}$$

initialized at some X_0, where $\{u_t, t \geq 1\}$ is a sequence of i.i.d. random variables with $\mathrm{E}u_1 = 0$ and $\mathrm{E}u_1^2 = \sigma^2$, $\{\rho_n, n \geq 1\}$ is a sequence of real numbers and $\{\phi_n, n \geq 1\}$ is a sequence of random variables. By setting $\mathcal{F}_{n,t} = \sigma(X_0, \phi_n, u_i, 1 \leq i \leq t)$ for $1 \leq t \leq n$ and $n \geq 1$, it follows from model (1.5) that

$$\mathrm{E}(X_t|\mathcal{F}_{n,t-1}) = (\rho_n + \phi_n)X_{t-1} \quad \text{and} \quad \mathrm{Var}(X_t|\mathcal{F}_{n,t-1}) = \sigma^2.$$

Thus the conditional LSE of ρ_n based on $X_0, \ldots, X_n$ is given by minimizing $\sum_{t=1}^n (X_t - \mathrm{E}(X_t|\mathcal{F}_{n,t-1}))^2$, i.e.,

$$\hat{\rho}_n = \frac{\sum_{t=1}^n X_{t-1}X_t}{\sum_{t=1}^n X_{t-1}^2} - \phi_n$$

and

$$\hat{\rho}_n - \rho_n = \frac{\sum_{t=1}^n X_{t-1}u_t}{\sum_{t=1}^n X_{t-1}^2}.$$

By imposing some conditions on the random coefficients and the initial value of X_0, Zhang and Yang [22] got the following result.

Theorem A. *As for the model (1.5) defined as above, suppose that the following two conditions are satisfied:*

(C1) $\{\phi_n, n \geq 1\}$ *is a sequence of random variables, independent of* $\{X_0, u_i, 1 \leq i \leq n\}$, *and*

$$|\rho_n + \phi_n| < 1, \qquad \mathrm{E}\left(\frac{1}{n(1-(\rho_n+\phi_n)^2)}\right) \to 0 \quad \text{as } n \to \infty;$$

(C2) X_0 *is independent of* $\{\phi_n, u_i, 1 \leq i \leq n\}$ *and* $\mathrm{E}X_0^2 = o(n)$.