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Staggered-grid spectral element methods for elastic wave simulations

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ABSTRACT

In this paper, we develop and analyze a new class of spectral element methods for the simulations of elastic wave propagation. The major components of the method are the spatial discretization and the choice of interpolation nodes. The spatial discretization is based on piecewise polynomial approximation defined on staggered grids. The resulting method combines the advantages of both staggered-grid based methods and classical non-staggered-grid based spectral element methods. Our new method is energy conserving and does not require the use of any numerical flux, because of the staggered local continuity of the basis functions. Our new method also uses Radau points as interpolation nodes, and the resulting mass matrix is diagonal, thus time marching is explicit and is very efficient. Moreover, we give a rigorous proof for the optimal convergence of the method. In terms of dispersion, we present a numerical study for the numerical dispersion and show that this error is of very high order. Finally, some numerical convergence tests and applications to unbounded domain problems with perfectly matched layer are shown.

1. Introduction

In this paper, we will develop and mathematically analyze a new class of spectral element methods for the elastic wave propagation. The major components of the method are the spatial discretization and the choice of interpolation nodes. The spatial discretization is based on piecewise polynomial approximation defined on staggered grids. The resulting method combines the advantages of both staggered-grid based methods and classical non-staggered-grid based spectral element methods. We summarize below the main advantages of the proposed staggered spectral element method

(1) high order accurate,
(2) low dispersion error,
(3) optimal convergence,
(4) conservation of energy, and
(5) diagonal mass matrix.

We now describe the problem setting. Let Ω be a bounded domain in $\mathbb{R}^2$ and $T > 0$ be a fixed time. Consider the following elastic wave equation

$$\rho \frac{\partial^2 u_x}{\partial t^2} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} + f_x,$$

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$$\rho\frac{\partial^2 u_z}{\partial t^2}=\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{zz}}{\partial z}+f_z,$$
$$\tau_{xx}=(\lambda+2\mu)\frac{\partial u_x}{\partial x}+\lambda\frac{\partial u_z}{\partial z},$$
$$\tau_{zz}=\lambda\frac{\partial u_x}{\partial x}+(\lambda+2\mu)\frac{\partial u_z}{\partial z},$$
$$\tau_{xz}=\mu\left(\frac{\partial u_x}{\partial z}+\frac{\partial u_z}{\partial x}\right).$$

In the above equations, $\vec{u}=(u_x,u_z)^t$ is the displacement vector, $\vec{\tau}=(\tau_{xx},\tau_{zz},\tau_{xz})^t$ is the stress tensor and $\vec{f}=(f_x,f_z)^t$ is a given source term. The function $\rho(x,z)$ is the density and $\lambda(x,z),\ \mu(x,z)$ are the Lamé coefficients. The above system is transformed into the following first-order hyperbolic system

$$\rho\frac{\partial v_x}{\partial t}=\frac{\partial \tau_{xx}}{\partial x}+\frac{\partial \tau_{xz}}{\partial z}+f_x, \tag{1}$$
$$\rho\frac{\partial v_z}{\partial t}=\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{zz}}{\partial z}+f_z, \tag{2}$$
$$L\frac{\partial \tau_{xx}}{\partial t}+M\frac{\partial \tau_{zz}}{\partial t}=\frac{\partial v_x}{\partial x}, \tag{3}$$
$$M\frac{\partial \tau_{xx}}{\partial t}+L\frac{\partial \tau_{zz}}{\partial t}=\frac{\partial v_z}{\partial z}, \tag{4}$$
$$\frac{1}{\mu}\frac{\partial \tau_{xz}}{\partial t}=\frac{\partial v_x}{\partial z}+\frac{\partial v_z}{\partial x}, \tag{5}$$

where we have introduced a new variable $\vec{v}=(v_x,v_z)^t=\left(\frac{\partial u_x}{\partial t},\frac{\partial u_z}{\partial t}\right)^t$, which is the velocity field. In Eqs. (1)–(5), we have $L=\frac{\lambda+2\mu}{4\mu(\mu+\lambda)}$ and $M=-\frac{\lambda}{4\mu(\mu+\lambda)}$. The above problem is equipped with the homogeneous Dirichlet boundary condition $v_x=v_z=0$ on $\partial\Omega$.

The numerical discretization of (1)–(5) is important in many geophysical applications [1], and the use of this mixed formulation allows the velocity and the stress tensor to be directly computed by the numerical scheme. There are in literature many numerical schemes with various strength and weakness. In Virieux [2], a second order in space and time staggered grid finite difference method is proposed for the system (1)–(5). Later in Levander [3], a fourth order in space and second order in time staggered grid finite difference method is also developed for the system (1)–(5). The use of staggered grids give many advantages described above.

The use of discontinuous Galerkin methods and spectral element methods have become popular in the numerical discretization of partial differential equations. For example, Brezzi, Marini and Süli [4] developed a discontinuous Galerkin method for the first order hyperbolic system, Gittelson, Hiptmair and Perugia [5] for the Helmholtz equation and Cockburn and Shu [6] for the convection–diffusion equation. For an overview and introduction of the subject, see Hesthaven and Warburton [7] and Riviére [8]. For the static elastic problems, discontinuous Galerkin methods have been proposed in Soon, Cockburn and Stolarski [9] and Wihler [10]. For the time dependent elastic wave propagation (1)–(5), discontinuous Galerkin methods are proposed in Käser and Dumbser [11] and Basabe, Sen and Wheeler [12]. In particular, the method of Käser and Dumbser [11] uses classical discontinuous Galerkin discretization with the solution of a generalized Riemann-problem for the numerical fluxes, while the method of Basabe, Sen and Wheeler [12] is based on the interior penalty formulation. On the other hand, spectral element method has been first proposed for the elastic wave equations in Komatitsch and Tromp [13], where high order Lagrange type basis functions are used on hexagonal elements. With the use of Gauss–Lobatto–Legendre quadrature, the resulting method has a diagonal mass matrix, and is therefore very efficient. In terms of dispersion analysis, the works Basabe, Sen and Wheeler [12] and Seriani and Oliveira [14] analyze the dispersion errors for some discontinuous Galerkin methods and spectral element methods, and show that these methods perform well.

Our work is motivated by the following observation. It is known that staggered grid finite difference methods give energy conservation. Moreover, spectral element methods give high order methods with diagonal mass matrices. Therefore, it is desirable to combine these ideas, and develop spectral element methods based on staggered grids.

Recently, a new class of discontinuous Galerkin methods based on a novel type of staggered grid is introduced in Chung and Engquist [15,16] for the wave equations, in Chung and Lee [17] and Chung and Kim [18] for the curl–curl operator, in Chung, Ciarlet and Yu [19] for Maxwell's equations, in Chung and Lee [20] for the convection–diffusion equation, in Kim, Chung and Lee [21] for the Stokes system and in Chan and Chung [22] for the Burgers equation. Moreover, wave transmission problems in the interface between classical material and meta-material using this kind of method is proposed and analyzed in Chung and Ciarlet [23], and fast solvers have been developed in Chung, Kim and Widlund [24] and Kim, Chung and Lee [25, 26]. These methods have the advantages that the structures, such as energy and mass, arising from the partial differential equations are preserved. Moreover, for time-dependent problems, the resulting mass matrices are block diagonal, giving