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Matched interface and boundary method for elasticity interface problems

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ABSTRACT

Elasticity theory is an important component of continuum mechanics and has had widely spread applications in science and engineering. Material interfaces are ubiquitous in nature and man-made devices, and often give rise to discontinuous coefficients in the governing elasticity equations. In this work, the matched interface and boundary (MIB) method is developed to address elasticity interface problems. Linear elasticity theory for both isotropic homogeneous and inhomogeneous media is employed. In our approach, Lamé's parameters can have jumps across the interface and are allowed to be position dependent in modeling isotropic inhomogeneous material. Both strong discontinuity, i.e., discontinuous solution, and weak discontinuity, namely, discontinuous derivatives of the solution, are considered in the present study. In the proposed method, fictitious values are utilized so that the standard central finite different schemes can be employed regardless of the interface. Interface jump conditions are enforced on the interface, which in turn, accurately determines fictitious values. We design new MIB schemes to account for complex interface geometries. In particular, the cross derivatives in the elasticity equations are difficult to handle for complex interface geometries. We propose secondary fictitious values and construct geometry based interpolation schemes to overcome this difficulty. Numerous analytical examples are used to validate the accuracy, convergence and robustness of the present MIB method for elasticity interface problems with both small and large curvatures, strong and weak discontinuities, and constant and variable coefficients. Numerical tests indicate second order accuracy in both L_∞ and L_2 norms.

1. Introduction

Elasticity interface problems play significant roles in continuum mechanics in which elasticity theory and related governing partial differential equations (PDEs) are commonly employed to describe various material behaviors. For this class of problems, an interface description in the elasticity theory is indispensable whenever there are voids, pores, inclusions, dislocations, cracks or composite structures in materials [1–4]. Elasticity interface problems are particularly important in tissue engineering, biomedical science and biophysics [5–7]. In many situations, the interface is not static such as fluid–structure interfacial boundaries [8]. Discontinuities in material properties often occur over the interface. Mathematically, there are two types of discontinuities, namely, strong discontinuities and weak discontinuities. Strong discontinuities are referred to situations where the displacement has jumps across the interface. In contrast, weak discontinuities are concerned with

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jumps in the gradient of the displacement, whereas the displacement is still continuous. In the linear elasticity theory, the stress–strain relation is governed by the constitutive equations. For isotropic homogeneous material, constitutive equations can be determined with any two terms of bulk modulus, Young's modulus, Lamé's first parameter, shear modulus, Poisson's ratio, and P-wave modulus [9]. If these moduli are position dependent functions, related constitutive equations can be used to describe elasticity property of isotropic inhomogeneous media. In seismic wave equations, inhomogeneity is accounted by assuming Lamé's parameters to be a position dependent function [10]. This model is also used in the elasticity analysis of biomolecules [5–7].

The study of the analytical solution for elasticity interface problems dated back to Eshelby in 1950s [11,12]. Working on inclusion and inhomogeneity problems, Eshelby found that for an infinite and elastically isotropic system with an ellipsoidal inhomogeneity, the eigenstrain distribution is uniform inside the inhomogeneity when it is subjected to a uniformly applied stress [11,12]. Much progress has been made on this area in the past few decades. Recently, semianalytic approaches for finding stress tensors have been proposed for arbitrarily shaped inhomogeneity [13].

Computationally, elasticity interface problems are more difficult than the corresponding Poisson interface problems because of the vector equation and cross derivatives. However, many numerical methods have been designed for elasticity interface problems. Based on meshes used, these methods can be classified into two types, i.e., algorithms relied on body-fitting meshes and algorithms based on special interface schemes. For the first type, meshes are generated to fit to the geometry of the interface without cutting through the interface. Therefore, adaptive meshes with local refinement techniques are frequently employed [14]. In the second type of algorithms, meshes are allowed to cut through the interface and particular schemes are designed to incorporate the interface information into the element shape function or discretization scheme. Immersed interface method (IIM) [15] has been used to solve elasticity interface problems for isotropic homogeneous media [16,17]. In this finite difference based algorithm, a local optimization scheme is designed for irregular grid points and the final linear equation with a non-symmetric matrix is solved by special solvers like BICG or GRMES. Second order accuracy is obtained [16]. A second-order sharp numerical method has been developed for linear elasticity equations [18]. Finite element based methods are also proposed for elasticity interface problems. Among them, the partition of unity method (PUM), the generalized finite element method (GFEM) and extended finite element method (XFEM) are developed to capture the non-smooth property of the solution over the interface by adding enrichment functions to the approximation [3,4,2]. Through the weak enforcement of the continuity, discontinuous Galerkin based methods have been employed to simulate strong and weak discontinuities [19–21]. Recently, immerse finite element (IFM) method has been proposed to solve elasticity problems with inhomogeneous jump conditions [22–24]. In this approach, finite element basis functions are adjusted locally to satisfy the jump conditions across the interface. Sharp-edged interface is considered for a special elasticity interface problem [25]. Lin, Sheen and Zhang have proposed a bilinear IFM and further modified it to a locking-free version [26,27]. For both compressible and nearly incompressible media, this method works well and offers second order accuracy. Recently, immersed meshfree Galerkin method has also been proposed for composite solids [28]. Most recently, a Nitsche type method has been proposed for elasticity interface problems [29]. Given the importance of elasticity interface problems in science and engineering, it is expected that more efficient numerical methods will be developed for this class of problems in the near future.

The matched interface and boundary (MIB) method was originally developed for solving Maxwell's equations [30] and elliptic interface problems [31–35]. A unique feature of the MIB method is that it provides a systematic procedure to achieve arbitrarily high order convergence for simple interfaces [30,33] and second order accuracy for arbitrarily complex interface geometry [31,32]. The essential idea is to introduce fictitious values at irregular mesh points which form fictitious domains [36] so that standard finite difference schemes can still be used across the interface. The lowest order interface jump conditions are iteratively enforced at the interface which determines fictitious values on fictitious domains. Typically, whenever possible, a high-dimensional interface problem is split into simple one-dimensional (1D) interface problems, similar to our earlier discrete singular convolution algorithm [36]. Due to the great flexibility in the construction of fictitious approximations, the MIB method has been shown to deliver up to 16th order accuracy for simple interfaces [30,33] and robust second order accuracy for arbitrarily complex interface geometry with geometric singularities (i.e., non-smooth interfaces with Lipschitz continuity) [31,32] and singular sources [35]. In the past decades, MIB method has been applied to a variety of problems. In computational biophysics, an MIB based Poisson–Boltzmann solver, MIBPB [37], has been constructed for the analysis of the electrostatic potential of biomolecules [31,35,38], molecular dynamics [39] and charge transport phenomenon [40,41]. Zhao has developed robust MIB schemes for the Helmholtz problems [42,43]. A second order accurate MIB method is constructed by Zhou and coworkers to solve the Navier–Stokes equations with discontinuous viscosity and density [44]. Recently, the MIB method has been used to solve elliptic equations with multi-material interfaces [45].

The objective of the present paper is to introduce the MIB method for solving elasticity interface problems. We consider both strong and weak discontinuities for isotropic homogeneous and inhomogeneous media. Computationally, the cross derivative terms in the elasticity model give rise to a new challenge for the MIB method when the interface geometry is complex. To overcome this difficulty, we modify the traditional fictitious definition and redefine fictitious values. With the MIB dimension splitting technique, a new fictitious representation is generated for each irregular mesh point based on elastic jump conditions and local geometry. Secondary fictitious values are constructed by the interpolation of these fictitious values and function values. We have designed schemes to deal with both small curvature and large curvature for complex interface geometries. To validate our method, analytical tests for different types of discontinuities and interface geometries are constructed. We demonstrate the second order accuracy of our MIB schemes for elasticity interface problems.