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Letter to the editor

A note on the error estimation of the Mann iteration ☆

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ABSTRACT

In this note, a new formula of error estimation of the Mann iteration and a sufficient condition of strong convergence of strongly demicontractive mappings are obtained. Moreover, some numerical examples are also given. Our results improve and extend the corresponding results in a recent article (L. Maruster and St. Maruster, 2015).

1. Introduction and preliminaries

Let C be a closed convex subset of real Hilbert space X and $\|\cdot\|$ denotes the norm on X. Let $T : C \to C$ and $F(T)$ denotes the set of fixed points of T. A sequence $\{x_n\}$ is generated by the Mann iteration of T if for $x_0 \in C$,

$$x_{n+1} = (1 - t_n)x_n + t_n Tx_n,$$

where $t_n \in [0, 1]$ and $n \geq 0$. Error estimations of Mann iteration for some contractive type mappings were considered by many authors [1–3]. Recently, L. Maruster and St. Maruster [4] first introduced a concept of strongly demicontractive mapping as follows:

The mapping T is said to be strongly demicontractive if $F(T) \neq \emptyset$ and

$$\|Tx - p\|^2 \leq a\|x - p\|^2 + K\|Tx - x\|^2, \quad \forall (x, p) \in C \times F(T),$$

where $a \in (0, 1)$ and $K \geq 0$ (if T is strongly demicontractive, then the fixed point is unique). And then they provided a posteriori error estimation of Mann iteration for strongly demicontractive mappings and obtained a condition of strong convergence of the Mann iteration.

Theorem 1.1 ([4, Corollary 1]). *Suppose that T satisfies the conditions of Theorem 1 in [4] and that it is asymptotically regular, i.e. $\|T^{(n+1)}x_0 - T^{(n)}x_0\| \to 0$ for some point x_0. Then the sequence $\{x_n\}$ generated by the Mann iteration with control sequence satisfying (3.3) in [4] and starting point x_0 converges strongly to the fixed point of T.*

Example 1.1 ([4, Example 1]). Let $C = [0.5, 1.5]$ and let T be a function $T : [0.5, 1.5] \to [-0.5, 1.5]$ defined by

$$Tx = \begin{cases} -x + 2 - (x - 1)^4 & \text{if } x < 0. \\ -x + 2 + (x - 1)^3 & \text{if } x \geq 0. \end{cases}$$

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This function is demicontractive with $a = 0.15,\ K = 0.25$. If we take $\theta_1 = 0.16,\ \theta_2 = 0.23$, then $t \in [0.906, 0.988]$ ($t_n \equiv t$) and

$$\frac{\|TT_t x - T_t x\|^2}{\|Tx - x\|^2} \geq 2\theta_2, \quad \forall x \in [0.5, 1.5].$$

Hence all conditions of Theorem 1 in [4] are satisfied.

Remark 1.1. Some comments (about Theorem 1.1 and Example 1.1) and an example are shown in the Appendix.

Inspired and motivated by the results in [4], we consider a new formula of error estimation and strong convergence of the Mann iteration for strongly demicontractive mappings.

2. New error estimation and convergence theorem

Lemma 2.1 ([2,4]). *Let $\{d_n\},\ \{\epsilon_n\}$ be nonnegative sequences of real numbers satisfying*

$$d_{n+1} \leq \alpha d_n + \epsilon_n$$

for all $n \geq 0$ and $0 \leq \alpha < 1$. If $\lim_{n\to\infty} \epsilon_n = 0$, then $\lim_{n\to\infty} d_n = 0$.

Lemma 2.2. *Let $\{d_n\}$ be nonnegative sequence satisfying*

$$d_{n+1} \leq \alpha d_n + \beta \epsilon_n \tag{2.1}$$

where $0 < \alpha < 1$, $\beta > 0$ and $\{\epsilon_n\}$ is a nonnegative sequence that satisfies the condition

$$\frac{\epsilon_{n+1}}{\epsilon_n} \leq \frac{\alpha}{2}, \quad \forall n \in \mathbb{N}. \tag{2.2}$$

Then $\lim_{n\to\infty} d_n = 0$ and

$$d_{n+1} \leq d_0 \alpha^{n+1} + \beta(\alpha^n \epsilon_0 + 2\alpha^{n-1}\epsilon_1).$$

Proof. By (2.2) and $0 < \alpha < 1$, we know that $\lim_{n\to\infty} \epsilon_n = 0$. It follows from Lemma 2.1 and the inequality (2.1) that $\lim_{n\to\infty} d_n = 0$. From the inequality (2.1), we can get

$$d_{n+1} \leq d_0 \alpha^{n+1} + \beta \sum_{i=0}^{n} \epsilon_i \alpha^{n-i}.$$

By (2.2) and induction on n, we have

$$\begin{aligned}
\sum_{i=0}^{n} \epsilon_i \alpha^{n-i} &= \alpha^n \epsilon_0 + \alpha^{n-1}\epsilon_1 + \cdots + \alpha^2 \epsilon_{n-2} + \alpha \epsilon_{n-1} + \epsilon_n \\
&\leq \alpha^n \epsilon_0 + \alpha^{n-1}\epsilon_1 + \cdots + \alpha^2 \epsilon_{n-2} + \left(1 + \frac{1}{2}\right) \alpha \epsilon_{n-1} \\
&\leq \alpha^n \epsilon_0 + \alpha^{n-1}\epsilon_1 + \cdots + \left(1 + \frac{1}{2} + \frac{1}{2^2}\right) \alpha^2 \epsilon_{n-2} \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\leq \alpha^n \epsilon_0 + \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}\right) \alpha^{n-1}\epsilon_1 \leq \alpha^n \epsilon_0 + 2\alpha^{n-1}\epsilon_1.
\end{aligned}$$

Therefore

$$d_{n+1} \leq d_0 \alpha^{n+1} + \beta(\alpha^n \epsilon_0 + 2\alpha^{n-1}\epsilon_1). \quad \square$$

Theorem 2.1. *Let T be a strongly demicontractive mapping with $0 < a < 1$ and $K \geq 0$. Assume that there exist positive numbers θ_1, θ_2, $0 < \theta_1 \leq \theta_2 < \min\{1,\ 1 - (1-a)(1-K)\}$ such that*

$$\frac{\|TT_{t_n} x - T_{t_n} x\|^2}{\|Tx - x\|^2} \leq \frac{\theta_2}{2}$$

where $x \in C$ and $T_{t_n} := (1 - t_n)I + t_n T$. For $x_0 \in C$, the sequence $\{x_n\}$ is generated by Mann iteration with the control sequence $\{t_n\}$ satisfying

$$\frac{1 - \theta_2}{1 - a} \leq t_n \leq \frac{1 - \theta_1}{1 - a}.$$