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Phase structure and asymptotic zero densities of orthogonal polynomials in the cubic model

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ABSTRACT

We apply the method we have described in a previous paper (2013) to determine the phase structure of asymptotic zero densities of the standard cubic model of non-Hermitian orthogonal polynomials. We provide a complete description of the two phases: the one cut phase and the two cut phase, and analyze the phase transition processes of the types: splitting of a cut, birth and death of a cut.

1. Introduction

Orthogonal polynomials in which the weight can be a complex function and the integration path can be a general curve in the complex plane (non-Hermitian orthogonal polynomials) first appeared in the mathematical literature as denominators of Padé and other types or rational approximants [1–4]. The theory quickly developed and found applications into such fields as the Riemann–Hilbert approach to strong asymptotics, random matrix theory [5–10] and in the study of dualities between supersymmetric gauge theories and string models [11–15]. This paper is devoted to a detailed analysis of the phase structure and phase transitions of the asymptotic (in the limit $n \to \infty$) zero density of monic orthogonal polynomials $P_n(z) = z^n + \cdots$

$$\int_{\Gamma} P_n(z) z^k \mathrm{e}^{-nW(z)} \mathrm{d}z = 0, \quad k = 0, \ldots, n-1, \tag{1}$$

where

$$W(z) = \frac{z^3}{3} - tz, \tag{2}$$

and t is an arbitrary complex number. The curve Γ is an infinite path connecting any two of the three different sectors shaded in Fig. 1 on which $\mathrm{Re}(z^3) > 0$, so that the integral in (1) is convergent and nonzero. Since the three possible choices are trivially related, without loss of generality we choose Γ as the curve connecting the two sectors with $\mathrm{Re}(z) < 0$.

The particular case $t = 0$ of this model has been rigorously studied by Deaño, Huybrechs and Kuijlaars [16], and recent results for $t \in \mathbb{R}$ have been communicated by Lejon [17]. The phase structure of the corresponding random matrix model has been studied by David [18] and Mariño [11].

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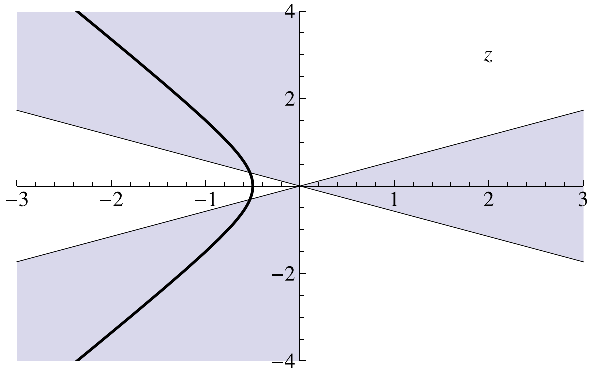

Fig. 1. The three regions of the complex z plane with $\mathrm{Re}(z^3) > 0$ and the curve Γ.

To perform our analysis we follow the method described in [19], which is applicable to study orthogonal polynomials with respect to general potentials

$$W(z) = \sum_{k=1}^{N} t_k z^k, \tag{3}$$

and appropriate choices of Γ. This method is based on the existence of a unique normalized equilibrium charge density that minimizes the electrostatic energy (among all normalized charge densities supported on the curve Γ) in the presence of the external electrostatic potential $V(z) = \mathrm{Re}\, W(z)$ [20]. The method in [19] also relies heavily on the concepts of S-property and S-curve of Stahl [1–4] and of Gonchar and Rakhmanov [21–23], and on the fundamental result of Gonchar and Rakhmanov [21] asserting that if Γ is an S-curve, then the asymptotic zero density of $P_n(z)$ exists and is given by the equilibrium charge density of the associated electrostatic model. Note that the integral (1) is invariant under deformations of the curve Γ into curves in the same homology class and connecting the same two convergence sectors at infinity. Recent results of Rakhmanov [23] and of Kuijlaars and Silva [24] show that for any family of orthogonal polynomials of the form (1) there always exists a deformation of Γ into an appropriate S-curve. We will make extensive use of certain algebraic curves (called *spectral curves*) which have the form

$$y^2 = W'(z)^2 + f(z), \tag{4}$$

where $f(z)$ is a polynomial of degree $\deg f = \deg W - 2$. More concretely, the main parameters that determine the S-curves and the associated equilibrium densities are the branch points of $y(z)$, which turn out to be the endpoints of the (in general, several disjoint) arcs that support the equilibrium density. The corresponding cuts are characterized as Stokes lines of the polynomial $y(z)^2$ or, equivalently, as trajectories of the quadratic differential $-y(z)^2(\mathrm{d}z)^2$. At this point numerical analysis will be necessary not only to solve the equations for the endpoints but also to analyze the existence of cuts satisfying the S-property.

The paper is organized as follows. In Section 2 we recall the main ingredients of the method presented in [19], namely, the relation between the electrostatic problem and the asymptotic zero density of the orthogonal polynomials, the S-property, and the notion of spectral curve. In Section 3 we determine the region of the complex t-plane in which the asymptotic density of zeros of the orthogonal polynomials is supported on a single cut (the one-cut phase of the cubic model (2)), and for each t in this region we calculate the corresponding S-curve and zero density. Since the number of disjoint arcs on which the asymptotic density of zeros is supported is less than or equal to $\deg W - 1$, the complement of the one-cut phase in the complex t-plane must be the region wherein the asymptotic density of zeros is supported on two disjoint arcs (the two-cut phase); we study this case in Section 4. Section 5 is dedicated to the study of phase transitions. We characterize critical processes of splitting, birth and death at a distance of cuts. The consistency of our results is checked in Section 6 by superimposing the cuts and the zeros of the corresponding orthogonal polynomials $P_n(z)$ with degree $n = 36$.

2. Zero densities of orthogonal polynomials and spectral curves

According to the general theory of logarithmic potentials with external fields [20], given an analytic curve Γ in the complex plane and a real-valued external potential $V(z)$, there exists a unique charge density $\rho(z)$ that minimizes the total electrostatic energy

$$\mathcal{E}[\rho] = \int_\Gamma |\mathrm{d}z| \rho(z) V(z) - \int_\Gamma |\mathrm{d}z| \int_\Gamma |\mathrm{d}z'| \log|z - z'| \rho(z)\rho(z') \tag{5}$$