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Convergence of Hermite interpolants on the circle using two derivatives[☆]

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In this paper we deal with Hermite interpolation problems on the unit circle considering up to the second derivative for the interpolation conditions and taking equally spaced points as nodal system. In the extended Fejér case, which corresponds to take vanishing values for the first two derivatives, we prove the uniform convergence for the interpolants related to continuous functions with smooth modulus of continuity. We also consider the Hermite case with non vanishing conditions for the derivatives for which we establish sufficient conditions on the interpolation conditions to obtain convergence.

1. Introduction

Classical Hermite interpolation problems on the unit circle and their generalizations have been studied by several researchers during the last years (see for example [1–5]). The nodes are usually equally spaced points and, in the standard case, the interpolation conditions correspond to the values of the polynomial and its first derivative. For the so called Hermite–Fejér case, a result generalizing Fejér's theorem for Chebyshev nodes, (see [6]), was obtained in [1]. Several results concerning the convergence of the interpolants were obtained in [7]. It is well known that similar problems for the interval $[-1, 1]$ were thoroughly studied in many classical papers (see [8] and [9]). On the unit circle sufficient conditions about the norms of the interpolation conditions when dealing with non vanishing derivatives were deduced in [10], in order to assure the convergence. Moreover the preceding conditions cannot be improved.

Our aim in this paper is to study generalized Hermite interpolation problems on the circle with equally spaced nodes and considering up to the second derivative. Useful expressions for computing these polynomials were obtained in [11]. Indeed, in [11], it was obtained barycentric expressions as well as expressions given in terms of orthogonal basis of subspaces of Laurent polynomials with respect to discrete Sobolev inner products connected with the nodal system.

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The purpose of the present paper is to obtain results about convergence for these generalized Hermite interpolation problems with equally spaced nodal systems on the circle. First we consider the generalized Hermite–Fejér case, that is, the first and second derivatives are zero at the nodes. In this situation we obtain uniform convergence for continuous functions with modulus of continuity satisfying a smooth condition. This result is more restrictive than in the classical case considering only the first derivative (see [1]). When we consider non vanishing derivatives, we also obtain convergence if we assume that the norms of the interpolation conditions satisfy some smooth properties. Now the results are similar to those obtained in the classical case (see [7]). Our construction includes a study of the rate of convergence of Hermite–Fejér interpolants for polynomial functions and a Brutman type theorem (see [12] and [13]). Notice that in the related problems of Hermite interpolation of higher order on bounded intervals and on trigonometric interpolation problems, most of the researchers consider smooth functions as can be seen in [14–16] and [17]; the corresponding Hermite–Fejér result on bounded interval can be seen in [18].

The organization of the paper is the following. In Section 2 we recall a preliminary result concerning the expression of the interpolation polynomials of which convergence will be studied in the paper. Section 3 is devoted to the study of the convergence of the Hermite–Fejér generalized interpolants for continuous functions. For this study we need some technical lemmas like an adaptation of classical Markov's result and an adaptation of Jackson's theorem that we present. In the last section we study the Hermite general case and we prove some results about convergence for a class of continuous functions. Indeed, we establish sufficient conditions on the interpolation conditions to obtain convergence.

2. Preliminaries

In the sequel we consider Hermite interpolation problems on the unit circle $\mathbb{T}$ using the first two derivatives. The nodal system $\{\alpha_j\}_{j=0}^{n-1}$ is constituted by the n-roots of a complex number λ, with $|\lambda| = 1$.

If $p(n)$ and $q(n)$ are two nondecreasing sequences of nonnegative integers such that $p(n) + q(n) = 3n - 1$ for $n \geq 1$ and $p(n) - q(n)$ is bounded we consider the unique Laurent polynomial $\mathcal{H}_{-p(n),q(n)}(z) \in \Lambda_{-p(n),q(n)}[z] = \operatorname{span}\{z^k : -p(n) \leq k \leq q(n)\}$ satisfying the interpolation conditions

$$\mathcal{H}_{-p(n),q(n)}(\alpha_j) = u_j, \qquad \mathcal{H}'_{-p(n),q(n)}(\alpha_j) = v_j, \quad \text{and} \quad \mathcal{H}''_{-p(n),q(n)}(\alpha_j) = w_j \quad j = 0, \dots, n-1, \tag{1}$$

where $\{u_j\}_{j=0}^{n-1}$, $\{v_j\}_{j=0}^{n-1}$ and $\{w_j\}_{j=0}^{n-1}$ are prefixed complex numbers. Although it is clear that α_j, u_j, v_j and w_j depend on n, however, for simplicity we omit the parameter n.

For simplicity and without loss of generality we consider the case corresponding to $p(n) = n + E[\frac{n}{2}]$ and $q(n) = n + E[\frac{n-1}{2}]$, where $E[x]$ denotes the integer part of x. Really we can obtain analogous results about convergence in the general case posed before.

It is well known that the interpolation polynomial $\mathcal{H}_{-n-E[\frac{n}{2}],n+E[\frac{n-1}{2}]}(z)$ can be written in terms of the fundamental polynomials of Hermite interpolation $\mathcal{A}_{j,n}(z)$, $\mathcal{B}_{j,n}(z)$ and $\mathcal{C}_{j,n}(z)$, for $j = 0, \dots, n-1$, as follows:

$$\mathcal{H}_{-n-E[\frac{n}{2}],n+E[\frac{n-1}{2}]}(z) = \sum_{j=0}^{n-1} (u_j \mathcal{A}_{j,n}(z) + v_j \mathcal{B}_{j,n}(z) + w_j \mathcal{C}_{j,n}(z)), \tag{2}$$

where $\mathcal{A}_{j,n}(z)$, $\mathcal{B}_{j,n}(z)$ and $\mathcal{C}_{j,n}(z)$ $(j = 0, \dots, n-1)$ are characterized by satisfying the following conditions

$$\begin{aligned}
&\mathcal{A}_{j,n}(\alpha_i) = \delta_{i,j}, \qquad \mathcal{A}'_{j,n}(\alpha_i) = 0, \qquad \mathcal{A}''_{j,n}(\alpha_i) = 0, \quad \forall i = 0, \dots, n-1, \\
&\mathcal{B}_{j,n}(\alpha_i) = 0, \qquad \mathcal{B}'_{j,n}(\alpha_i) = \delta_{i,j}, \qquad \mathcal{B}''_{j,n}(\alpha_i) = 0, \quad \forall i = 0, \dots, n-1, \\
&\mathcal{C}_{j,n}(\alpha_i) = 0, \qquad \mathcal{C}'_{j,n}(\alpha_i) = 0, \qquad \mathcal{C}''_{j,n}(\alpha_i) = \delta_{i,j}, \quad \forall i = 0, \dots, n-1.
\end{aligned}$$

It can be seen in [11] suitable expressions for these fundamental polynomials. Next we recall these expressions distinguishing two cases according to whether the number of nodal points is even or odd.

If we assume that the nodal system has $2n$ points, then it can be seen in [11] that the fundamental polynomials of the Hermite interpolation in the Laurent space $\Lambda_{-3n,3n-1}[z]$, $\mathcal{A}_{j,2n}(z)$, $\mathcal{B}_{j,2n}(z)$ and $\mathcal{C}_{j,2n}(z)$, for $j = 0, \dots, 2n-1$, have the following expressions

$$\begin{aligned}
\mathcal{A}_{j,2n}(z) &= \frac{\alpha_j(1-n^2)}{16\lambda\alpha_j^n n^3} \frac{\left(z^{2n}-\lambda\right)^3}{z^{3n}\left(z-\alpha_j\right)} + \frac{3\alpha_j^2}{16\lambda\alpha_j^n n^3} \frac{\left(z^{2n}-\lambda\right)^3}{z^{3n}\left(z-\alpha_j\right)^2} + \frac{\alpha_j^3}{8\lambda\alpha_j^n n^3} \frac{\left(z^{2n}-\lambda\right)^3}{z^{3n}\left(z-\alpha_j\right)^3}, \\
\mathcal{B}_{j,2n}(z) &= \frac{3\alpha_j^2}{16\lambda\alpha_j^n n^3} \frac{\left(z^{2n}-\lambda\right)^3}{z^{3n}\left(z-\alpha_j\right)} + \frac{\alpha_j^3}{8\lambda\alpha_j^n n^3} \frac{\left(z^{2n}-\lambda\right)^3}{z^{3n}\left(z-\alpha_j\right)^2}, \\
\mathcal{C}_{j,2n}(z) &= \frac{\alpha_j^3}{16\lambda\alpha_j^n n^3} \frac{\left(z^{2n}-\lambda\right)^3}{z^{3n}\left(z-\alpha_j\right)}.
\end{aligned} \tag{3}$$