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Multiple orthogonal polynomials on the unit circle. Normality and recurrence relations

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To Pablo González-Vera, our beloved master and friend. In memoriam

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A B S T R A C T

Multiple orthogonal polynomials on the unit circle (MOPUC) were introduced by J. Mínguez and W. Van Assche for the first time in 2008. Some applications were given there and recurrence relations were obtained from a Riemann–Hilbert problem.

This paper is a second contribution to this field. We first obtain a determinantal formula for MOPUC (multiple Heine's formula) and we analyze the concept of normality, from a dynamical point of view and by presenting a first example: the combination of the Lebesgue and Rogers–Szegő measures. Secondly, we deduce recurrence relations for MOPUC without using Riemann–Hilbert analysis, only by considering orthogonality conditions. This new approach allows us to complete the recurrence relations in the situation when the origin is a zero of MOPUC, a case that was not considered before. As a consequence, we give an appropriate definition of multiple Verblunsky coefficients. A multiple version of the well known Szegő recurrence relation is also obtained. Here, the coefficients that appear in the recurrence satisfy certain partial difference equations that are used to present a recursive algorithm for the computation of MOPUC. A discussion on the Riemann–Hilbert approach that also includes the case when the origin is a zero of MOPUC is presented. Some conclusions and open questions are finally mentioned.

1. Introduction and OPUC

This paper is devoted to the study of Multiple Orthogonal Polynomials on the Unit Circle (from now on, MOPUC), specially from the point of view of their recurrence relations and their recursive computation.

Despite what occurs in the case where the supports of the measures are on the real axis (see e.g. [1]), the situation when dealing with measures supported on the unit circle is barely studied up to now, to the extent that it is possible to mention a unique previous reference in this sense [2]. Therefore, the renewed interest in recent years by the orthogonal polynomials in the Unit Circle (OPUC, also called Szegő polynomials), specially after the publication of the monograph [3], demands further study on MOPUC. In particular, in [2] it is pointed out the usefulness of such polynomials both for two-point Hermite–Padé approximation to Carathéodory functions (playing a similar role as their counterparts in the real axis with respect to the Markov functions) and for linear prediction in multivariate time series. In addition, we are also interested in the study of simultaneous quadrature rules in the unit circle (see e.g. [4–6], among others, for the real case).

In the real case, in addition to the above mentioned monograph by Nikishin and Sorokin [1], it is worth mentioning some seminal papers about multiple (or Hermite–Padé) orthogonal polynomials, such as [7–11], and more recently, [12–14], to cite only a few. For our purposes in the present paper, it is also remarkable that in [15] the characterization of orthogonal

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polynomials by means of a matrix Riemann–Hilbert problem (see [16]) was extended for multiple orthogonal polynomials and employed to get suitable recurrence relations. As it was said above, about MOPUC there is only a previous Ref. [2], where the Riemann–Hilbert analysis is also used to derive recurrence relations.

In the present paper, we are concerned with type II MOPUC, following the denomination coined for Multiple Orthogonal Polynomials on the Real Line. The reason of considering only type II MOPUC, and not Type I, is due to our further interest in studying simultaneous quadrature formulae in the unit circle.

Now, it is convenient to start with some convention for notation. We denote by $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ the unit circle of the complex plane and by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{E} := \{z \in \mathbb{C} : |z| > 1\}$ its interior and exterior, respectively. $\mathbb{P} := \mathbb{C}[z]$ is the complex vector space of polynomials in the variable z with complex coefficients and $\mathbb{P}_n := \text{span}\{1, z, \ldots, z^n\}$ the corresponding vector subspace of polynomials with degree less than or equal to n. For a given polynomial $P_n \in \mathbb{P}_n \backslash \mathbb{P}_{n-1}$ we define its reversed (or reciprocal) as $P_n^*(z) = z^n \overline{P_n(1/\bar{z})} \in \mathbb{P}_n$. Also, the vector space of square matrices with complex coefficients and dimension n will be denoted by $\mathcal{M}_n$.

Let μ be a positive measure with support on $\mathbb{T}$. We consider the Hilbert space $L_2^\mu(\mathbb{T})$ of measurable functions ψ for which $\int_{-\pi}^{\pi} \left|\psi(e^{i\theta})\right|^2 d\mu(\theta) < +\infty$ and the inner product induced by μ is given by $\langle \phi, \varphi \rangle_\mu = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(z)\overline{\varphi(z)} d\mu(\theta)$, where $\phi, \varphi \in L_2^\mu(\mathbb{T})$ and $z = e^{i\theta}$. Observe that $\langle z^n f, g \rangle_\mu = \langle f, z^{-n} g \rangle_\mu$. We denote by $\{\rho_n\}_{n=0}^\infty$ the sequence of monic orthogonal polynomials on the unit circle (OPUC) for μ, or Szegő polynomials (see [17, Chapter 11]). Namely, setting $\rho_0 = \rho_0^* \equiv 1$, for each $n \geq 1$, ρ_n is a monic polynomial of exact degree n satisfying

$$\langle \rho_n(z), z^s \rangle_\mu = \langle \rho_n^*(z), z^t \rangle_\mu = 0 \quad \text{for all } s = 0, 1, \ldots, n-1, \; t = 1, 2, \ldots, n,$$
$$\|\rho_n\|_\mu^2 = \langle \rho_n(z), z^n \rangle_\mu = \|\rho_n^*\|_\mu^2 = \langle \rho_n^*(z), 1 \rangle_\mu > 0.$$

It is very well known that the polynomials ρ_n and ρ_n^* have all its zeros on $\mathbb{D}$ and $\mathbb{E}$, respectively (see [17, Theorem 11.4.1]).

For our purposes in Section 3, let us recall a way to obtain the well known Szegő recurrence relation (see [17, Theorem 11.4.2]). The monic polynomial $z\rho_{n-1} \in \mathbb{P}_n \backslash \mathbb{P}_{n-1}$ is orthogonal to $\text{span}\{z, \ldots, z^{n-1}\}$ (as the polynomial $\rho_{n-1}^* \in \mathbb{P}_{n-1}$). If occasionally $\langle z\rho_{n-1}, 1 \rangle_\mu = 0$, then it must hold $\rho_n = z\rho_{n-1}$. On the contrary, if we set $R_n := z\rho_{n-1} + \delta_n \rho_{n-1}^* \in \mathbb{P}_n \backslash \mathbb{P}_{n-1}$, with δ_n an arbitrary constant, then we can choose it so that $\langle R_n, 1 \rangle_\mu = 0$. More precisely, when

$$\delta_n = -\frac{\langle z\rho_{n-1}, 1 \rangle_\mu}{\langle \rho_{n-1}^*, 1 \rangle_\mu}, \tag{1}$$

(which is well defined and it is different from zero in this case), then it must hold $R_n = \rho_n$. In general, in the computation of ρ_n we need ρ_{n-1} and ρ_{n-1}^*, so it is often to find this recurrence written in matrix form as

$$\begin{pmatrix} \rho_n \\ \rho_n^* \end{pmatrix} = \begin{pmatrix} z & \delta_n \\ \overline{\delta_n} z & 1 \end{pmatrix} \begin{pmatrix} \rho_{n-1} \\ \rho_{n-1}^* \end{pmatrix}, \quad n = 1, 2, \ldots. \tag{2}$$

The parameters $\delta_0 = 1$ and $\delta_n = \rho_n(0) \in \mathbb{D}$ for all $n \geq 1$ are called the reflection, Schur or Verblunsky coefficients associated with μ, see [3, Chapter 1.5].

Let us consider the Hermitian sequence of trigonometric moments $e_k := \langle z^k, 1 \rangle_\mu$, $e_{-k} = \overline{e_k}$ for $k \geq 0$, that we assume that are known in advance. If we have computed ρ_{n-1} in the iteration $n-1$, from $\{e_k\}_{k=-n}^n$ we obtain δ_n from (1) and hence we can compute ρ_n from (2). We notice also that from (2) we can deduce the following three-term recurrence relation for Szegő polynomials that is only valid if the Verblunsky coefficient δ_{n-1} does not vanish:

$$\rho_n(z) = \left(\frac{\delta_n}{\delta_{n-1}} + z \right) \rho_{n-1}(z) - \frac{\delta_n}{\delta_{n-1}} \left(1 - |\delta_{n-1}|^2\right) z\rho_{n-2}(z). \tag{3}$$

A determinantal formula for Szegő and their reversed polynomials is very well known. Indeed, if Δ_n denotes the nth Toeplitz matrix for μ i.e.,

$$\Delta_n := \begin{pmatrix} e_0 & e_1 & \cdots & e_n \\ e_{-1} & e_0 & \cdots & e_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ e_{-n} & e_{-n+1} & \cdots & e_0 \end{pmatrix} \in \mathcal{M}_{n+1}, \quad (\det \Delta_n \neq 0), \; n \geq 0, \tag{4}$$

then it holds (Heine's formula)

$$\rho_n(z) = \frac{1}{\det \Delta_{n-1}} \begin{vmatrix} e_0 & e_1 & \cdots & e_n \\ e_{-1} & e_0 & \cdots & e_{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ e_{-n+1} & e_{-n+2} & \cdots & e_1 \\ 1 & z & \cdots & z^n \end{vmatrix}, \quad n \geq 1, \tag{5}$$