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Orthogonal polynomials for Minkowski's question mark function

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ABSTRACT

Hermann Minkowski introduced a function in 1904 which maps quadratic irrational numbers to rational numbers and this function is now known as Minkowski's question mark function since Minkowski used the notation $?(x)$. This function is a distribution function on $[0, 1]$ which defines a singular continuous measure with support $[0, 1]$. Our interest is in the (monic) orthogonal polynomials $(P_n)_{n\in\mathbb{N}}$ for the Minkowski measure and in particular in the behavior of the recurrence coefficients of the three term recurrence relation. We will give some numerical experiments using the discretized Stieltjes–Gautschi method with a discrete measure supported on the Minkowski sequence. We also explain how one can compute the moments of the Minkowski measure and compute the recurrence coefficients using the Chebyshev algorithm.

1. Introduction

In 1904 Hermann Minkowski [23] introduced an interesting function, which he called the question mark function and he denoted its values by $?(x)$. This notation with a question mark is somewhat confusing, so instead we will denote the function by q and we will only consider it on the interval $[0, 1]$.

There are several ways to define the Minkowski question mark function. Minkowski used the following construction: let $\mathcal{M}_1$ be the sequence with two elements 0 and 1 and define $q(0) = 0$ and $q(1) = 1$. The sequence $\mathcal{M}_2$ then consists of $\mathcal{M}_1$ and the new point $(0 + 1)/(1 + 1) = 1/2$ and $q(1/2) = 1/2$. In general we construct the *Minkowski sequence* $\mathcal{M}_N$ by taking all the elements from $\mathcal{M}_{N-1}$ and all the "*mediants*" $(a+a')/(b+b')$ of two consecutive rational numbers a/b and a'/b' in $\mathcal{M}_{N-1}$, where we take $0 = 0/1$ and $1 = 1/1$. Then the Minkowski question mark function on the new points takes the values

$$q\left(\frac{a+a'}{b+b'}\right) = \frac{q(a/b) + q(a'/b')}{2}.$$

The Minkowski sequence $\mathcal{M}_N$ is dense in $[0, 1]$ as $N \to \infty$ and $q(x)$ for $x \in [0, 1] \setminus \mathbb{Q}$ is defined by continuity. Observe that $\mathcal{M}_N$ contains $2^{N-1} + 1$ points.

Another way to define the question mark function is by using continued fractions [13]. If $0 < x < 1$ then we can write x as a regular continued fraction

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}}, \quad a_i \in \mathbb{N} \setminus \{0\}.$$

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The Minkowski question mark function at x is then defined as

$$q(x) = 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^{a_1+a_2+\cdots+a_k}}.$$

If x is a rational number, then the continued fraction is terminating and $q(x)$ is given by a finite sum. By setting $q(0) = 0$ and $q(1) = 1$ one can show that $q : [0, 1] \to [0, 1]$ is a continuous and increasing function, so that q is a probability distribution function on [0, 1] which defines a probability measure on [0, 1]. Arnaud Denjoy [14] showed that this distribution function has the property that $q'(x) = 0$ almost everywhere on [0, 1] so that the corresponding measure is singular and continuous.

A third way is to define the question mark function as a fixed point of an iterated function system with two rational functions. One has

$$q(x) = \begin{cases} \frac{1}{2} q\left(\frac{x}{1-x}\right), & 0 \le x \le \frac{1}{2}, \\ 1 - \frac{1}{2} q\left(\frac{1-x}{x}\right), & \frac{1}{2} < x \le 1, \end{cases} \tag{1}$$

and one can easily show that the sequence of probability distribution functions $(q_n)_{n\in\mathbb{N}}$, with

$$q_n(x) = \begin{cases} \frac{1}{2} q_{n-1}\left(\frac{x}{1-x}\right), & 0 \le x \le \frac{1}{2}, \\ 1 - \frac{1}{2} q_{n-1}\left(\frac{1-x}{x}\right), & \frac{1}{2} < x \le 1, \end{cases}$$

and q_0 any probability distribution on [0, 1], converges uniformly to Minkowski's question mark function. We have used this method to plot the question mark function in Fig. 1. This allows us to compute integrals by a limit procedure

$$\int_0^1 f(x)\, dq(x) = \lim_{n\to\infty} \int_0^1 f(x)\, dq_n(x).$$

In 1943 Raphaël Salem posed a problem about the Fourier coefficients of Minkowski's question mark function:

$$\alpha_n = \int_0^1 e^{2in\pi x}\, dq(x).$$

The Riemann–Lebesgue lemma tells us that Fourier coefficients of an absolutely continuous measure on [0, 1] tend to zero. The Minkowski question mark function is singularly continuous, so one cannot use the Riemann–Lebesgue lemma. Nevertheless, the support of q is the full interval [0, 1] and it was proved by Salem [27] that q is Hölder continuous of order $\alpha = \log 2/(2\log \frac{\sqrt{5}+1}{2}) = 0.7202$. Furthermore, Salem showed that

$$\frac{1}{n}\sum_{k=0}^{n} |\alpha_k| = \mathcal{O}(n^{-\alpha/2}),$$

so that α_n converges to zero on the average and there is the possibility that $\alpha_n \to 0$. This is the problem posed by Raphaël Salem [27]: do the Fourier coefficients of the Minkowski question mark function converge to 0? This is still an open problem. Giedrius Alkauskas [1,2] already investigated this extensively by both numerical and analytical methods.

Our interest in this paper is in the orthonormal polynomials for the Minkowski question mark function:

$$\int_0^1 p_n(x)p_m(x)\, dq(x) = \delta_{m,n},$$

where $p_n(x) = \gamma_n x^n + \cdots$ and $\gamma_n > 0$, with recurrence relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_np_n(x) + a_np_{n-1}(x), \quad n \ge 0, \tag{2}$$

with $p_0 = 1$ and $p_{-1} = 0$, and in particular we are interested in the asymptotic behavior of the recurrence coefficients $(a_n)_{n\ge1}$ and $(b_n)_{n\ge0}$. Rakhmanov's theorem [25,26] tells us that for an absolutely continuous measure μ on [0, 1] for which $\mu' > 0$ almost everywhere on [0, 1], one has $a_n \to 1/4$ and $b_n \to 1/2$ as $n \to \infty$. In our case $q' = 0$ almost everywhere, so one cannot use Rakhmanov's theorem to deduce the asymptotic behavior of the recurrence coefficients. However, it is known (see, e.g., [21,32,29]) that there exist discrete measures and continuous singular measures on [0, 1] for which the recurrence coefficients have the behavior $b_n \to 1/2$ and $a_n \to 1/4$ as $n \to \infty$, so that they are in the Nevai class $M(\frac{1}{2}, \frac{1}{4})$.

Definition 1. The Nevai class $M(b, a)$ consists of all positive measures on the real line for which the orthogonal polynomials have recurrence coefficients satisfying

$$\lim_{n\to\infty} a_n = a, \qquad \lim_{n\to\infty} b_n = b.$$

It is well known that measures $\mu \in M(b, a)$ have essential spectrum $[b - 2a, b + 2a]$, i.e., the support of μ is $[b - 2a, b + 2a] \cup E$, where E is at most countable and the accumulation points can only be at $b \pm 2a$ (Blumenthal's theorem, see, e.g., [24, Thm. 7 on p. 23], [31, Section 5]).