Journal of Computational and Applied Mathematics 284 (2015) 235–243

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Generalized anti-Gauss quadrature rules[☆]

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ARTICLE INFO

Article history:
Received 11 August 2014
Received in revised form 11 November 2014

In memory of Pablo González Vera

MSC:
primary 65D30
65D32
65F15
secondary 41A55

Keywords:
Gauss quadrature
Anti-Gauss quadrature
Error estimate

ABSTRACT

Gauss quadrature is a popular approach to approximate the value of a desired integral determined by a measure with support on the real axis. Laurie proposed an $(n+1)$-point quadrature rule that gives an error of the same magnitude and of opposite sign as the associated n-point Gauss quadrature rule for all polynomials of degree up to $2n+1$. This rule is referred to as an anti-Gauss rule. It is useful for the estimation of the error in the approximation of the desired integral furnished by the n-point Gauss rule. This paper describes a modification of the $(n+1)$-point anti-Gauss rule, that has $n+k$ nodes and gives an error of the same magnitude and of opposite sign as the associated n-point Gauss quadrature rule for all polynomials of degree up to $2n+2k-1$ for some $k > 1$. We refer to this rule as a generalized anti-Gauss rule. An application to error estimation of matrix functionals is presented.

1. Introduction

Let $d\mu$ be a nonnegative measure with infinitely many points of support such that the integral

$$\mathcal{I}f := \int_a^b f(x)d\mu(x), \quad -\infty \le a < b \le \infty, \tag{1.1}$$

is defined for all polynomials f. We assume that all moments

$$\mu_j := \int_a^b x^j d\mu(x), \quad j = 0, 1, 2, \ldots,$$

exist and that the measure is normalized so that $\mu_0 = 1$. Introduce the inner product

$$(f, g) := \mathcal{I}(fg) \tag{1.2}$$

for polynomials f and g, and let $\{p_j\}_{j=0,1,2,\ldots}$ denote the sequence of monic orthogonal polynomials with respect to this inner product, i.e.,

$$(p_j, p_k) \begin{cases} > 0, & j = k, \\ = 0, & j \ne k, \end{cases}$$

where p_j is of degree j with leading coefficient one.

☆ Version November 11, 2014.

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http://dx.doi.org/10.1016/j.cam.2014.11.016
0377-0427/

Let the function f be continuous on the convex hull of the support of the measure $d\mu$. The n-point Gauss quadrature rule associated with $d\mu$ is of the form

$$\mathcal{G}_n f := \sum_{i=1}^{n} f(x_i) w_i \tag{1.3}$$

and is characterized by the property that

$$\mathcal{I} f = \mathcal{G}_n f \quad \forall f \in \mathbb{P}_{2n-1}, \tag{1.4}$$

where $\mathbb{P}_{2n-1}$ denotes the set of all polynomials of degree at most $2n-1$.

The orthogonal polynomials p_j satisfy a three-term recursion relation

$$\begin{aligned} p_1(x) &= (x-a_0)p_0(x), \quad p_0(x) = 1, \\ p_{i+1}(x) &= (x-a_i)p_i(x) - b_i p_{i-1}(x), \quad i = 1, 2, \ldots, \end{aligned} \tag{1.5}$$

with $b_i > 0$. The recursion relations for $p_0, p_1, \ldots, p_n$ can be expressed as

$$x \begin{bmatrix} p_0(x) \\ p_1(x) \\ \vdots \\ p_{n-1}(x) \end{bmatrix} = J_n \begin{bmatrix} p_0(x) \\ p_1(x) \\ \vdots \\ p_{n-1}(x) \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ p_n(x) \end{bmatrix},$$

where

$$J_n = \begin{bmatrix} a_0 & 1 & & & 0 \\ b_1 & a_1 & 1 & & \\ & & & \ddots & \\ & & \ddots & \ddots & 1 \\ 0 & & & b_{n-1} & a_{n-1} \end{bmatrix} \in \mathbb{R}^{n \times n}. \tag{1.6}$$

We note that the matrix J_n can be symmetrized by a real diagonal similarity transformation. Denote the symmetrized tridiagonal matrix by T_n. It is well known that the eigenvalues and the squares of the first components of the normalized eigenvectors of T_n are the nodes and weights of the Gauss rule (1.3). They can be computed efficiently with the Golub–Welsch algorithm (see, e.g., [1–3]) or by a method described by Laurie [4].

This paper is concerned with the following $(n+k)$-point quadrature rules for $k \geq 2$. They generalize the $(n+1)$-point anti-Gauss rule introduced by Laurie [5], which is obtained when $k = 1$.

Definition 1.1. The generalized anti-Gauss quadrature rule

$$\widetilde{\mathcal{G}}_{n+k}^{(k)} f := \sum_{i=1}^{n+k} f(\widetilde{x}_i^{(k)}) \widetilde{w}_i^{(k)} \tag{1.7}$$

is an $(n+k)$-point quadrature rule such that

$$(\mathcal{I} - \widetilde{\mathcal{G}}_{n+k}^{(k)}) f = -(\mathcal{I} - \mathcal{G}_n) f \quad \forall f \in \mathbb{P}_{2n+2k-1}. \tag{1.8}$$

It follows from (1.4) that

$$\widetilde{\mathcal{G}}_{n+k}^{(k)} f = \mathcal{I} f \quad \forall f \in \mathbb{P}_{2n-1}. \tag{1.9}$$

For notational simplicity, we assume that the nodes are ordered according to

$$\widetilde{x}_1^{(k)} < \widetilde{x}_2^{(k)} < \cdots < \widetilde{x}_{n+k}^{(k)}.$$

We can express (1.8) as

$$\widetilde{\mathcal{G}}_{n+k}^{(k)} f = (2\mathcal{I} - \mathcal{G}_n) f \quad \forall f \in \mathbb{P}_{2n+2k-1}, \tag{1.10}$$

which shows that when the functional $\widetilde{\mathcal{G}}_{n+k}^{(k)}$ exists, we may consider it an $(n+k)$-point Gauss quadrature rule associated with the functional $2\mathcal{I} - \mathcal{G}_n$. The latter functional is said to be quasi-definite if every leading $k \times k$ principal submatrix (for $k = 1, 2, \ldots$) of the infinite Hankel matrix

$$H = \begin{bmatrix} \widetilde{\mu}_0 & \widetilde{\mu}_1 & \widetilde{\mu}_2 & \widetilde{\mu}_3 & \cdots \\ \widetilde{\mu}_1 & \widetilde{\mu}_2 & \widetilde{\mu}_3 & & \\ \widetilde{\mu}_2 & \widetilde{\mu}_3 & & & \\ \widetilde{\mu}_3 & & & & \\ \vdots & & & & \ddots \end{bmatrix}$$