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Spectral properties of birth–death polynomials

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ABSTRACT

We consider sequences of polynomials that are defined by a three-terms recurrence relation and orthogonal with respect to a positive measure on the nonnegative axis. By a famous result of Karlin and McGregor such sequences are instrumental in the analysis of birth–death processes. Inspired by problems and results in this stochastic setting we present necessary and sufficient conditions in terms of the parameters in the recurrence relation for the smallest or second smallest point in the support of the orthogonalizing measure to be larger than zero, and for the support to be discrete with no finite limit point.

1. Introduction

We are concerned with a sequence of polynomials $\{P_n\}$ defined by the three-terms recurrence relation

$$\begin{aligned} &P_{n+1}(x) = (x - \lambda_n - \mu_n)P_n(x) - \lambda_{n-1}\mu_n P_{n-1}(x), \quad n > 0, \\ &P_1(x) = x - \lambda_0 - \mu_0, \quad P_0(x) = 1, \end{aligned} \tag{1}$$

where $\lambda_n > 0$ for $n \geq 0$, $\mu_n > 0$ for $n \geq 1$ and $\mu_0 \geq 0$. Since polynomial sequences of this type play an important role in the analysis of *birth–death processes* – continuous-time Markov chains on an ordered set with transitions only to neighbouring states – we will refer to $\{P_n\}$ as the sequence of *birth–death polynomials* associated with the *birth rates* λ_n and *death rates* μ_n. For more information on the relation between a sequence of birth–death polynomials and the corresponding birth–death process we refer to the seminal papers of Karlin and McGregor [1,2].

By Favard's theorem (see, for example, Chihara [3]) there exists a probability measure (a Borel measure of total mass 1) on $\mathbb{R}$ with respect to which the polynomials P_n are orthogonal. In the terminology of the theory of moments the *Hamburger moment problem* associated with the polynomials P_n is solvable. Actually, as shown by Karlin and McGregor [1] and Chihara [4] (see also [3, Theorem I.9.1 and Corollary]), even the *Stieltjes moment problem* associated with $\{P_n\}$ is solvable, which means that there exists an orthogonalizing measure ψ for $\{P_n\}$ with support on the nonnegative axis, that is,

$$\int_{[0,\infty)} P_n(x)P_m(x)\psi(dx) = k_n\delta_{nm}, \quad n, m \geq 0, \tag{2}$$

with $k_n > 0$. The Stieltjes moment problem associated with $\{P_n\}$ is said to be *determined* if ψ is uniquely determined by (2), and *indeterminate* otherwise. In the latter case there is, by [5, Theorem 5], a unique orthogonalizing measure for which

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the infimum of its support is maximal. We will refer to this measure as the *natural* measure for $\{P_n\}$. In what follows ψ will always refer to the natural measure for $\{P_n\}$ if the Stieltjes moment problem associated with $\{P_n\}$ is indeterminate.

Of particular interest to us are the quantities ξ_i, recurrently defined by

$$\xi_1 := \inf \mathrm{supp}(\psi), \tag{3}$$

and

$$\xi_{i+1} := \inf\{\mathrm{supp}(\psi) \cap (\xi_i, \infty)\}, \quad i \geq 1, \tag{4}$$

where $\mathrm{supp}(\psi)$ denotes the support of the measure ψ, also referred to as the *spectrum* of ψ (or of the polynomials P_n). In addition we let

$$\sigma := \lim_{i\to\infty} \xi_i, \tag{5}$$

the first limit point of $\mathrm{supp}(\psi)$ if it exists, and infinity otherwise. It is clear from the definition of ξ_i that, for all $i \geq 1$,

$$\xi_{i+1} \geq \xi_i \geq 0,$$

and

$$\xi_i = \xi_{i+1} \Longleftrightarrow \xi_i = \sigma.$$

In the analysis of a birth–death process on a countable state space – a birth–death process on the nonnegative integers with birth rate λ_n and death rate μ_n in state n, say – the question of whether the time-dependent transition probabilities of the process converge to their limiting values exponentially fast as time goes to infinity has attracted considerable attention. This question may be translated into the setting of the polynomials P_n of (1) by asking whether $\xi_1 > 0$, and if not, whether $\xi_2 > 0$, since the exponential rate of convergence (or *decay parameter*) α of the birth–death process satisfies

$$\alpha = \begin{cases} \xi_1 & \text{if } \xi_1 > 0 \\ \xi_2 & \text{if } \xi_2 > \xi_1 = 0 \\ 0 & \text{if } \xi_2 = \xi_1 = 0 \end{cases}$$

(see, for example, [6]). Note that

$$\alpha > 0 \Longleftrightarrow 0 < \sigma \leq \infty, \tag{6}$$

so the above question may be rephrased by asking whether $0 < \sigma \leq \infty$. Recent results, in particular in the Chinese literature, have culminated in a complete solution of the problem in the stochastic setting by revealing simple and easily verifiable conditions for exponential convergence in terms of the birth and death rates. The purpose of this paper is to present these results in an orthogonal-polynomial context, and to provide new proofs for some of the results by using tools from the orthogonal-polynomial toolbox. Our methods enable us also to establish a simple, necessary and sufficient condition for $\sigma = \infty$, that is, for the spectrum of the orthogonalizing measure to be discrete with no finite limit point, thus extending another recent result.

Before stating the results in Section 3 and discussing proofs in Section 4 we present a number of preliminary results in Section 2. Additional information on related literature and some concluding remarks will be given in Section 5.

2. Preliminaries

It will be convenient to define the constants π_n by

$$\pi_0 := 1 \quad \text{and} \quad \pi_n := \frac{\lambda_0\lambda_1 \dots \lambda_{n-1}}{\mu_1\mu_2 \dots \mu_n}, \quad n > 0, \tag{7}$$

and to use the shorthand notation

$$K_n := \sum_{i=0}^{n} \pi_i, \quad n \geq 0, \qquad K_\infty := \sum_{i=0}^{\infty} \pi_i, \tag{8}$$

and

$$L_n := \sum_{i=0}^{n} (\lambda_i\pi_i)^{-1}, \quad n \geq 0, \qquad L_\infty := \sum_{i=0}^{\infty} (\lambda_i\pi_i)^{-1}. \tag{9}$$

With the convention that the measure ψ in (2) is interpreted as the natural measure if the Stieltjes moment problem associated with $\{P_n\}$ is indeterminate, the quantities ξ_i and σ of (3)–(5) may be defined alternatively in terms of the (simple and positive) zeros of the polynomials $P_n(x)$ (see [3, Section II.4]). Namely, with $x_{n1} < x_{n2} < \cdots < x_{nn}$ denoting the n zeros of $P_n(x)$, we have the classic separation result

$$0 < x_{n+1,i} < x_{ni} < x_{n+1,i+1}, \quad i = 1, 2, \dots, n, \; n \geq 1,$$