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Optimal interval estimation for a family of lower truncated distribution under progressive censoring

Liang Wang

School of Mathematics and Statistics, Xidian University, Xi'an 710071, PR China

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ABSTRACT

In this paper, based on progressively Type-II censored sample, different types of interval estimates of a general lower truncated distribution are constructed. Exact confidence intervals and joint exact confidence regions are derived for unknown model parameter and the lower truncated threshold bound. Optimal criteria are also provided for choosing the best confidence interval and confidence region among obtained estimates. Two real-life examples and a numerical study are presented to illustrate the performance of our results.

1. Introduction

The application of truncated distributions arises in a wide variety of practical areas such as reliability engineering, survival analysis, lifetime study among others, where the range of a random variable is limited to values which lie above or below a specified threshold or within a specified range. For instance, studies of income based on incomes above or below some poverty line may be of limited usefulness for inference about the whole population, then these parts of the data are ignored. Another case to employ truncated distribution is in description on the development of the pit depths on a water pipe where the pit depth is ranged in an interval $[0, h_0]$ with h_0 being the thickness of the pipe (see Sheikh et al. [1]). Several authors have studied the parameter estimation of truncated distribution and corresponding application. The characteristics and application of the truncated Weibull distribution are discussed by Zhang and Xie [2], where parametric analysis and parameter estimation methods of the distribution are investigated. Aban et al. [3] studied parameter estimation methods for the truncated Pareto distribution. Ruiz and Navarro [4] obtained the distribution function of conditional expectations for a double truncated model. Nadarajah [5] presented some truncated long-tailed distributions, of which the explicit expressions for the moments are also derived.

For arbitrary continuous lifetime model with cumulative distribution function (CDF) $F(x)$ and the probability density function (PDF) $f(x)$, the double truncated distribution of $F(x)$ can be expressed as

$$F_{DT}(x) = \frac{F(x) - F(a)}{F(b) - F(a)} \quad \text{and} \quad f_{DT}(x) = \frac{f(x)}{F(b) - F(a)}, \quad a \le x \le b, \tag{1}$$

where a and b are lower and upper truncated bounds, respectively. Note that, when $a = 0$ and $b \to \infty$, it corresponds to origin distribution $F(x)$. When $a \to 0$, it reduces to the upper truncated distribution and when $b \to \infty$, it is the lower truncated distribution.

Let X be a random variable from a family of continuous distribution with CDF

$$F(x; \theta) = 1 - [1 - G(x)]^{\theta}, \quad \theta > 0, x > 0, \tag{2}$$

E-mail address: wangl@xidian.edu.cn.

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where $\theta > 0$ is unknown parameter, $G(x)$ is an arbitrary continuous distribution function, free of unknown parameters with $G(0) = 0$ and $G(\infty) = 1$. The family in (2) is well-known in lifetime experiments as proportional hazard rate (PHR) model (see e.g. Lawless [6]), includes several well-known lifetime distribution such as exponential, Burr type XII, Lomax, Resnick, Weibull (one parameter) among others, and the family is a subclass of a regular one parameter exponential family of distributions. Inferences for this model have been studied by several authors, see for example Ahmadi et al. [7] and Wang and Shi [8].

The corresponding PDF of (2) can be derived as

$$f(x;\theta) = \theta g(x)[1 - G(x)]^{\theta-1}, \quad \theta > 0, x > 0, \tag{3}$$

where $g(x) = (d/dx)G(x)$ is the density function of $G(x)$.

Based on (1)–(3), let $b \to \infty$, one can obtain the lower truncated PHR distribution with CDF and PDF as

$$F_{LT}(x) = 1 - \left[\frac{1-G(x)}{1-G(a)}\right]^{\theta} \quad \text{and} \quad f_{LT}(x) = \frac{\theta g(x)}{1-G(a)}\left[\frac{1-G(x)}{1-G(a)}\right]^{\theta-1}, \quad x \geq a. \tag{4}$$

Some motivations should be mentioned for the necessity of the truncated version of the PHR distribution. One most common reason is that, as the improvement of techniques, products especially for high reliability and long lifetime products feature a long failure-free life period, which is particularly true in electronic industries where environmental stress screening is widely applied, resulting in long failure-free life after the final burn-in. Thus it should be more appropriate to use a truncated model to analyze failure data, where a truncated distribution gives a higher weight to the failure data. Another reason is that, the PHR distribution is a popular lifetime distribution. The feature of long failure-free life of modern products needs the inference for the truncated version of the PHR model where the truncated version also maintains the properties of the original distribution.

The survival function $R_{LT}(t)$ and hazard rate function $H_{LT}(t)$ of the lower PHR distribution at mission time $t \geq a$, can be written as

$$R_{LT}(t) = \left[\frac{1-G(t)}{1-G(a)}\right]^{\theta} \quad \text{and} \quad H_{LT}(t) = \frac{\theta g(t)}{1-G(t)}.$$

Censoring is very common phenomenon in many reliability experiments and lifetime studies because of the time limits and other restrictions on data collecting procedure. Generally speaking, censoring implies that exact failure times are known for only a portion of the units under study. There are several types of censored tests, the most common censored tests are Type I and Type II censoring, where n units are subjected to some form of test and the test is terminated after a fixed time or exactly $m(< n)$ units fail. However, in manufacturing process of modern industry, the product features high lifetime and high test cost, in order to save time and money in experiments, based on above censoring mechanism, progressively censored test were introduced in practice, namely progressively Type-I and Type-II censoring, which is a more efficient and economic method of obtaining data in lifetime studies. The property of progressively censoring with different lifetime distributions has been studied extensively by many authors. Some works, for example, can be found in Ali Mousa and Jaheen [9], Fernández [10], Wu [11], Pradhan and Kundu [12], Viveros and Balakrishnan [13]. For more details about progressively censoring, see Balakrishnan and Aggarwala [14]. Since the season that progressive censored test is an efficient method to obtain failure data of modern products, especially for long lifetime and high reliability products, and that lower truncated PHR distribution may provide a better fit than original model for available data with long failure-free lifetime products, this motivates us to discuss the inference problem for the truncated PHR model when failure data is progressively censored.

The progressive Type-II censoring can be described in following way: suppose that n independent units are put in a life test and the censoring scheme $R = (r_1, r_2, \ldots, r_m)$ is previously fixed. When first failure time, say $X_{1:m:n}$, has occurred, r_1 surviving units are randomly removed from the $n-1$ remaining surviving units. When second failure time, say $X_{2:m:n}$, has occurred, r_2 surviving units are randomly removed from the $n - r_1 - 2$ remaining units. Proceeding the procedure until the mth failure, say $X_{m:m:n}$, all the remaining $r_m = n - m - r_1 - \cdots - r_{m-1}$ surviving units are removed. Then $X_{1:m:n} < X_{2:m:n} < \cdots < X_{m:m:n}$ are called progressively Type-II censored order statistics. From the conception of progressively Type-II censoring scheme, it is observed that several conventional test scheme are special cases of progressively Type-II censoring such as Type-II censoring, multiple Type-II and complete sample test.

In practice, although the lower truncated bound a exists, sometimes one cannot know its exact value in most cases. In this work we treat a as an unknown threshold parameter as same as the parameter θ, i.e., both θ and a are unknown in the lower truncated PHR model (4), and exact interval estimation is considered for both unknown parameters. The rest of this paper is organized as follows. Section 2 is devoted to propose different interval estimators for θ and a. Two real-life examples and a numerical illustration are provided in Section 3 to illustrate our methods. Section 4 presents some concluding remarks.

2. Interval estimation of model parameters

In this section, exact confidence intervals and confidence regions are proposed for unknown parameters θ and a, and some optimal criteria are also provided to find the optimal ones.