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Monodromy of a class of analytic generalized nilpotent systems through their Newton diagram

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ABSTRACT

Newton diagram of a planar vector field allows to determine whether a singular point of an analytic system is a monodromic singular point. We solve the monodromy problem for the nilpotent systems and we apply our method to a wide family of systems with a degenerate singular point, so-called generalized nilpotent cubic systems.

1. Introduction

Newton diagram is an important tool for studying singularities of maps and vector fields, see e.g. classical book [1]. We consider the planar analytic differential system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}), \tag{1}$$

with $\mathbf{F}(\mathbf{0}) = \mathbf{0}$, i.e. the origin is a singular point of the system, and we are interested in characterizing, through the Newton diagram of the vector field, when the origin is surrounded by orbits of the system, or, equivalently, we want to determine when the system does not have characteristic orbits at the origin, i.e. trajectories that enter or leave the origin with a fixed tangent. When this occurs, the singular point is called monodromic.

From the finiteness theorem for the number of limit cycles, a monodromic point of an analytic planar vector field can be only either a focus (all trajectory by lying on a vicinity of a monodromic singular point is a spiral) or a center (the trajectories are closed orbits that surround at the origin), see [2]. So, the monodromy problem is a preliminary to solve the center problem which is one of the classical open problems in the qualitative theory of planar differential systems.

In relation to the monodromy problem, unless the differential matrix $D\mathbf{F}(\mathbf{0})$ is not identically null, the problem was completely solved by Poincaré [3], and Andreev [4] for nilpotent systems. Finally, if $D\mathbf{F}(\mathbf{0})$ is identically null, the origin is a degenerate singular point, there are only a few partial results: Algaba et al. [5], García et al. [6], Gasull et al. [7], Mañosa [8] and Medvedeva [9], among others. Recently, Algaba et al. [10] give an algorithm based on the main result of [5] and as application they solve the monodromy problem for the family

$$\dot{x} = ay^3 + cxy^2 + gx^2y + ex^5, \qquad \dot{y} = dy^3 + hxy^2 + fx^4y + bx^7,$$

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with $ab(g^2 + h^2) > 0$, which extends to the family studied in [9]. We emphasize that all of them are families of polynomial systems.

Here, we deal with the analytic case. From [5, Theorem 3], if origin of (1) is monodromic, then the system is

$$\dot{x} = ay^{2n-1} + f(x, y), \qquad \dot{y} = bx^{2m-1} + g(x, y), \tag{2}$$

with $ab < 0$ and f and g are analytic functions satisfying $\partial f/\partial y(0, 0) = \partial^j f/\partial y^j(0, 0) = 0$, for $j = 1, \dots, 2n - 1$, and $\partial g/\partial x(0, 0) = \partial^j g/\partial x^j(0, 0) = 0$, for $j = 1, \dots, 2m - 1$. Moreover, we can assume that $n \le m$.

In this paper we solve the monodromy problem for the analytic nilpotent systems through the Newton diagram of the vector field, i.e. systems (2) with $n = 1$. This result is stated in Theorem 3. For systems (2) with $n = 2$, so-called generalized nilpotent cubic systems, Theorems 5–7 characterize when the lowest degree quasi-homogeneous term of vector field determine its monodromy. Otherwise, Propositions 9 and 10 establish when the systems with invariant axis $x = 0$ and whose Newton diagram has the inner vertex $(1, 2)$, do not have any characteristic orbits different from $x = 0$, which allows us to determine the monodromy of the generalized nilpotent cubic systems, after performing a blow-up. Theorem 8 summarizes this result.

In short, we solve the monodromy problem for the analytic systems (2) with $n = 1$ (solved before by Andreev) and $n = 2$. For $n \ge 3$, by using our techniques, it is possible to characterize the monodromy in the cases which the lowest-degree quasi-homogeneous term determines it. The remaining cases only we can provide necessary conditions of monodromy. In general, it is necessary the performing of a series of blows up to be able to determine the monodromy. So, in general, the problem remains open for analytic systems.

2. Preliminaries

To show our results, we recall some concepts, which we use throughout the paper.

Conservative–dissipative splitting.

Let $\mathbf{t} = (t_1, t_2)$ non-null with t_1 and t_2 non-negative integer numbers without common factors. A function f of two variables is a quasi-homogeneous function of type $\mathbf{t}$ and degree k if $f(\varepsilon^{t_1}x, \varepsilon^{t_2}y) = \varepsilon^k f(x, y)$. The vector space of quasi-homogeneous polynomials of type $\mathbf{t}$ and degree k will be denoted by $\mathcal{P}_k^{\mathbf{t}}$. A vector field $\mathbf{F} = (F_1, F_2)^T$ is quasi-homogeneous of type $\mathbf{t}$ and degree k if $F_1 \in \mathcal{P}_{k+t_1}^{\mathbf{t}}$ and $F_2 \in \mathcal{P}_{k+t_2}^{\mathbf{t}}$. We will denote by $\mathcal{Q}_k^{\mathbf{t}}$ the vector space of the quasi-homogeneous polynomial vector fields of type $\mathbf{t}$ and degree k.

The quasi-homogeneous vector monomials can be determined by drawing the lattice $\mathbb{Z}_+^2$, and assigning each point (m, n) to the quasi-homogeneous vector fields $(x^m y^{n-1}, 0)^T$ and $(0, x^{m-1}y^n)^T$. The points with integer coordinates aligned in the straight lines perpendicular to $\mathbf{t}$, $(m - 1)t_1 + (n - 1)t_2 = k$, determine the quasi-homogeneous vector monomials with the same degree k.

Any vector field can be expanded into quasi-homogeneous terms of type $\mathbf{t}$ of successive degrees. Thus, system (1) can be written as

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = \mathbf{F}_r^{\mathbf{t}}(\mathbf{x}) + \mathbf{F}_{r+1}^{\mathbf{t}}(\mathbf{x}) + \cdots = \sum_{j=0}^{\infty} \mathbf{F}_{r+j}^{\mathbf{t}}(\mathbf{x}),$$

for some $r \in \mathbb{Z}$, where $\mathbf{F}_j^{\mathbf{t}} = (P_{j+t_1}, Q_{j+t_2})^T \in \mathcal{Q}_j^{\mathbf{t}}$ and $\mathbf{F}_r^{\mathbf{t}} \not\equiv \mathbf{0}$. Such expansions are expressed as $\mathbf{F} = \mathbf{F}_r^{\mathbf{t}} +$ q-h.h.o.t.

Such expansions are valuable tools to analyze the singularity, see [11]. This concept also is used for the study of the integrability, the center problem and the existence of an inverse integrating factor of systems with a degenerate singular point, i.e. systems whose matrix of the linear part evaluated in the singular point is identically null, see [12–14].

Next, we show the splitting of a quasi-homogeneous vector field as a sum of two quasi-homogeneous vector fields, a conservative one (having zero-divergence) and a dissipative one that plays a main role in our analysis. Throughout this paper, the Hamiltonian system associated to the $\mathcal{C}^1$ function f is denoted by $\mathbf{X}_f$, i.e. $\mathbf{X}_f = (-\frac{\partial f}{\partial y}, \frac{\partial f}{\partial x})^T$. Algaba et al. [13] have proved that any quasi-homogeneous vector field $\mathbf{F}_j^{\mathbf{t}} = (P_{j+t_1}, Q_{j+t_2})^T \in \mathcal{Q}_j^{\mathbf{t}}$ can be expressed as

$$\mathbf{F}_j^{\mathbf{t}} = \mathbf{X}_{h_{j+|\mathbf{t}|}} + \mu_j \mathbf{D}_0^{\mathbf{t}}, \tag{3}$$

where $\mathbf{D}_0^{\mathbf{t}}(x, y) := (t_1 x, t_2 y)^T$ (a dissipative quasi-homogeneous vector field of type $\mathbf{t}$ and degree 0), $(j+|\mathbf{t}|)\mu_j := \mathrm{div}(\mathbf{F}_j^{\mathbf{t}}) \in \mathcal{P}_j^{\mathbf{t}}$ (divergence of $\mathbf{F}_j^{\mathbf{t}}$), $(j + |\mathbf{t}|)h_{j+|\mathbf{t}|} := t_1 x Q_{j+t_2} - t_2 y P_{j+t_1} \in \mathcal{P}_{j+|\mathbf{t}|}^{\mathbf{t}}$ (wedge product of $\mathbf{D}_0^{\mathbf{t}}$ and $\mathbf{F}_j^{\mathbf{t}}$) and $|\mathbf{t}| = t_1 + t_2$.

It is a simple matter to show that any non-vanishing quasi-homogeneous polynomial of type $\mathbf{t} = (t_1, t_2)$ with t_1 and t_2 non-null, in particular $h_{j+|\mathbf{t}|}$, can be expressed as $p(x, y) = x^{k_1} y^{k_2} p_0(x^{t_2}, y^{t_1})$ with $0 \le k_1 < t_2$, $0 \le k_2 < t_1$ being p_0 a homogeneous polynomial. So, by abusing the notation, it is possible to write any quasi-homogeneous polynomial of type $\mathbf{t}$ in a compact form $p(x, y) = c \prod_{j=0}^{m} f_j^{m_j} \prod_{j=0}^{n} g_j^{n_j}$, where

$$f_j(x, y) = x,\ y \quad \text{or} \quad y^{t_1} - \lambda_j x^{t_2}, \quad j = 0, \dots, m$$