Journal of Computational and Applied Mathematics 287 (2015) 98–114

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Phase portraits of uniform isochronous quartic centers

Jackson Itikawa, Jaume Llibre *

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

ARTICLE INFO

Article history:
Received 16 September 2014
Received in revised form 21 January 2015

MSC:
primary 34C05
34C25

Keywords:
Polynomial vector field
Uniform isochronous center
Phase portrait

ABSTRACT

In this paper we classify the global phase portraits in the Poincaré disc of all quartic polynomial differential systems with a uniform isochronous center at the origin such that their nonlinear part is not homogeneous.

© 2015 Elsevier B.V. All rights reserved.

1. Introduction and statement of the main results

Christian Huygens is credited with being one of the first scholars to study isochronous systems in the XVII century, even before the development of the differential calculus. Huygens investigated the cycloidal pendulum, which has isochronous oscillations in opposition to the monotonicity of the period of the usual pendulum. It is probably the first example of a nonlinear isochrone. For more details see [1].

Isochronicity appears in a wide variety of Physics phenomena and it is also closely related to the uniqueness and existence of solutions for some boundary value, perturbation, or bifurcation problems. Moreover it is important in stability theory, since a periodic solution in the region surrounding the center type singular point is Liapunov stable if and only if the neighboring periodic solutions have the same period. For more details on these topics see [2]. In the last decades the study of isochronous systems has been increased due to the proliferation of powerful methods of computerized research, and special attention has been dedicated to polynomial differential systems, see [3–6] and the bibliography therein.

In this paper we classify the global phase portraits of all quartic polynomial differential systems with a uniform isochronous center at the origin such that their nonlinear part is not homogeneous.

Let $p \in \mathbb{R}^2$ be a center of a differential polynomial system in $\mathbb{R}^2$, without loss of generality we can assume that p is the origin of coordinates. We say that p is an *isochronous center* if it is a center having a neighborhood such that all the periodic orbits in this neighborhood have the same period. We say that p is a *uniform isochronous center* if the system, in polar coordinates $x = r\cos\theta,\ y = r\sin\theta$, takes the form $\dot{r} = G(\theta, r), \dot{\theta} = k,\ k \in \mathbb{R} \setminus \{0\}$, for more details see Conti [5].

Proposition 1. *Assume that a planar differential polynomial system $\dot{x} = P(x, y),\ \dot{y} = Q(x, y)$ of degree n has a center at the origin of coordinates. Then, this center is uniform isochronous if and only if by doing a linear change of variables and a rescaling of time it can be written into the form*

$$\dot{x} = -y + x f(x, y), \qquad \dot{y} = x + y f(x, y), \tag{1}$$

where $f(x, y)$ is a polynomial in x and y of degree $n - 1$, and $f(0, 0) = 0$.

* Corresponding author. Tel.: +34 93 5811303; fax: +34 935812790.
E-mail addresses: itikawa@mat.uab.cat (J. Itikawa), jllibre@mat.uab.cat (J. Llibre).

http://dx.doi.org/10.1016/j.cam.2015.02.046
0377-0427/© 2015 Elsevier B.V. All rights reserved.

Since we cannot find a proof of the well known Proposition 1 in the literature we have provided a proof in our paper at the beginning of Section 3.

Algaba et al. [7] in 1999, and Chavarriga et al. [8] in 2001, independently provided the following characterization of quartic polynomial systems with an isolated uniform isochronous center at the origin.

Theorem 2. *Consider $f(x,y)=\sum_{i=1}^{3} f_i(x,y)$ where $f_i(x,y)$ for $i=1,2,3$ are homogeneous polynomials of degree i, $f_1^2+f_2^2\neq 0$ and $f_3\neq 0$ such that (1) be a quartic polynomial differential system. Then the only case of local analytic integrability in an open neighborhood of the origin of system (1) is given, modulo a rotation, by the time-reversible system.*

$$\begin{aligned}\dot{x}&=-y+x(A_1x+B_2xy+C_1x^3+C_3xy^2),\\ \dot{y}&=\ \ x+y(A_1x+B_2xy+C_1x^3+C_3xy^2),\end{aligned}\tag{2}$$

where $A_1, B_2, C_1, C_3\in\mathbb{R}$.

By the following classical result due to Poincaré [9] and Liapunov [10] Theorem 2 characterizes the quartic uniform isochronous centers, except the ones for which the polynomial $f(x,y)$ is a homogeneous polynomial of degree 3.

Theorem 3. *An analytic differential system $\dot{x}=-y+F_1(x,y)$, $\dot{y}=x+F_2(x,y)$, with $F_1(x,y)$ and $F_2(x,y)$ real analytic functions without constant and linear terms defined in a neighborhood of the origin, has a center at the origin if and only if there exists a local analytic first integral of the form $H=x^2+y^2+G(x,y)$ defined in a neighborhood of the origin, where G starts with terms of order higher than two.*

Algaba et al. [7] provided the phase portraits of systems (2) in the particular case $C_1=0$. In such case systems (2) have a polynomial commutator, allowing to get the bifurcation diagram of the systems. In Theorem 4, we provide all the global phase portraits of systems (2).

Theorem 4. *Consider a quartic polynomial differential system $X:\mathbb{R}^2\to\mathbb{R}^2$ and assume that X has a uniform isochronous center at the origin such that their nonlinear part is not homogeneous. Then the global phase portrait of X is topologically equivalent to one of the 14 phase portraits of Fig. 1.*

More precisely, since X can always be written as system (2), the global phase portrait of X is topologically equivalent to the phase portrait

(a) *of Fig. 1 if either $C_1C_3>0$, or $C_3=0,\ B_2<0$;*
(b) *of Fig. 1 if $C_1=0,\ C_3\neq 0$ and if either $r_1,r_2,r_3>0$, or $r_1,r_2,r_3<0$;*
(c) *of Fig. 1 if $C_1=0,\ C_3\neq 0$ and if either $r_1<0, r_2,r_3>0$, or $r_1,r_2<0, r_3>0$;*
(d) *of Fig. 1 if $C_1=0,\ C_3\neq 0$ and if either $r_1r_2>0, r_3=r_2$, or $r_2=r_1, r_1r_3>0$;*
(e) *of Fig. 1 if $C_1=0,\ C_3\neq 0$ and if either $r_1<0, r_2>0, r_3=r_2$, or $r_2=r_1, r_1<0, r_3>0$;*
(f) *of Fig. 1 if $C_1=0,\ C_3\neq 0$ and if either $r_3=r_2=r_1,\ \forall r_1,r_2,r_3\in\mathbb{R}^*$, or $r_1\neq 0$ and $r_{2,3}=a\pm bi,\ \forall r_1,b\in\mathbb{R}^*, a\in\mathbb{R}$;*
(g) *of Fig. 1 if $C_3=0,\ C_1\neq 0,\ B_2>0,\ C_1\neq -A_1B_2$;*
(h) *of Fig. 1 if either $C_3=0,\ C_1\neq 0,\ B_2>0,\ C_1=-A_1B_2$, or $B_2=C_3=0$;*
(i) *or* (j) *or* (k) *of Fig. 1 if $C_1C_3<0,\ B_2=0$;*
(j) *or* (m) *or* (n) *of Fig. 1 if $C_1C_3<0,\ B_2\neq 0$;*

where in the cases with $C_1=0$, we have that r_1,r_2,r_3 are the roots of the polynomial $-C_3-B_2x-A_1x^2-x^3$ and we assume that $r_1\le r_2\le r_3$ when these roots are real.

Our results have been checked with the software *P4*, see for more details on this software the Chapters 9 and 10 of [11].

The rest of the paper is organized as follows. In Section 2 we present some results and technical propositions used in our study. In Section 3 we prove Theorem 4.

2. Preliminary results

In this section we present some results necessary to our study.

Poincaré compactification

Let $\mathcal{X}$ be a planar vector field of degree n. The *Poincaré compactified vector field $p(\mathcal{X})$ corresponding to* $\mathcal{X}$ is an analytic vector field induced on $\mathbb{S}^2$ as follows (see, for instance [12], or Chapter 5 of [11]). Let $\mathbb{S}^2=\{y=(y_1,y_2,y_3)\in\mathbb{R}^3 : y_1^2+y_2^2+y_3^2=1\}$ (the *Poincaré sphere*) and $T_y\mathbb{S}^2$ be the tangent space to $\mathbb{S}^2$ at point y. Consider the central projection $f : T_{(0,0,1)}\mathbb{S}^2\to\mathbb{S}^2$. This map defines two copies of $\mathcal{X}$, one in the northern hemisphere and the other in the southern hemisphere. Denote by $\mathcal{X}'$ the vector field $Df\circ\mathcal{X}$ defined on $\mathbb{S}^2$ except on its equator $\mathbb{S}^1=\{y\in\mathbb{S}^2 : y_3=0\}$. Clearly $\mathbb{S}^1$ is identified to the *infinity* of $\mathbb{R}^2$. In order to extend $\mathcal{X}'$ to a vector field on $\mathbb{S}^2$ (including $\mathbb{S}^1$) it is necessary that $\mathcal{X}$ satisfies suitable conditions. In the case that $\mathcal{X}$ is a planar vector field of degree n then $p(\mathcal{X})$ is the only analytic extension of $y_3^{n-1}\mathcal{X}'$ to $\mathbb{S}^2$. On $\mathbb{S}^2\backslash\mathbb{S}^1$

Download English Version:

https://daneshyari.com/en/article/4638527

Download Persian Version:

https://daneshyari.com/article/4638527

Daneshyari.com