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Goal-oriented adaptive finite element methods for elliptic problems revisited

Markus Bürg [a], Murtazo Nazarov [b,*]

[a] Department of Mathematics, Texas A&M University, USA

[b] Division of Scientific Computing, Department of Information Technology, Uppsala University, Box 337, SE-751 05 Uppsala, Sweden

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ABSTRACT

A goal-oriented a posteriori error estimation of an output functional for elliptic problems is presented. Continuous finite element approximations are used in quadrilateral and triangular meshes. The algorithm is similar to the classical dual-weighted error estimation, however the dual weight contains solutions of the proposed patch problems. The patch problems are introduced to apply Clément and Scott–Zhang type interpolation operators to estimate point values with the finite element polynomials. The algorithm is shown to be reliable, efficient and convergent.

1. Introduction

In this paper we develop *a goal-oriented a posteriori* error estimation with respect to certain target functionals. A goal oriented adaptive finite element method has been an active research of many scientists since last three decades and goes back to earlier work of Erikson and Johnson, Becker and Rannacher, with co-workers, see [1–5]. The error in the target functional, or the so-called *quantity of interest* is written as a product of the residual of the underlying primal problem and the corresponding *adjoint* or *dual solution*. Although dual-weighted *a posteriori* error estimates are applied successfully for various problems and impressive performance was obtained in terms of efficiency and computability, see e.g. [6–8], the convergence of the adaptive algorithm was not known until the work of [9,10]. In [9] the dual-weighted term is kept element-wise, and by making rather stronger regularity assumptions for the primal and dual solutions they proved the convergence and optimality of the adaptive algorithm. Whereas in [10] product of the energy norms of primal and dual problems is kept globally. The estimator marks cells with respect to the energy norms of primal and dual problems separately, then the set of marked cells with the smallest cardinality is refined. The convergence and optimality of the algorithm are established for the scaled Poisson equation, however the generalization for more complex differential equations is not clear. In [11] the product of energy norms of the primal and dual problems is separated by Hölder's inequality, then the union of the sets with the largest error with respect to both primal and dual error indicators is chosen for the refinement. The approximated error is overestimated in this case, nevertheless the convergence of the underlying adaptive algorithm is obtained using the contraction framework by [12].

The presented method in this paper is closely related to the classical dual-weighted algorithm, i.e. the error on the quantity of interest is estimated by the sum of the cell errors, which are the product of the primal and dual contributions.

* Corresponding author.
E-mail address: murtazo.nazarov@it.uu.se (M. Nazarov).

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Moreover, finite element spaces for the primal and dual solutions can be the same. The main idea consists of using Clément or Scott–Zhang type interpolation operator to estimate the continuous dual solution by a local average of the underlying finite element space. First, we prove reliability and efficiency of the new algorithm. We then prove the convergence and show the optimality of the algorithm numerically. The proof of optimality of the algorithm is under investigations and will be reported in due time.

To avoid confusion in notations between triangular and quadrilateral elements, we develop the main framework and proofs for quadrilateral meshes. Nevertheless, the below analysis also apply to triangular meshes.

The paper is organized as follows. In Section 2 we give the standard finite element notations and the problem formulation. Section 3 is the main contribution of this paper. We introduce an adaptive algorithm based on goal-oriented a posteriori error estimation, we prove its reliability, efficiency. Then in Section 3 we discuss convergence of the proposed adaptive algorithm for h and hp refinements. Number of numerical illustrations is given in Section 5 to support the theory presented in this paper.

2. Preliminaries

In this section, we want to fix some notations and introduce the basic assumptions which we require throughout this work. Further, the elliptic model problem is presented and we introduce the basic idea of goal-oriented adaptivity.

2.1. Notations and basic assumptions

Let $\Omega \subset \mathbb{R}^2$ denote some open and bounded domain. We denote the Lebesgue space of square-integrable functions in Ω by $L^2(\Omega)$ and its dual by $L^2(\Omega)'$. The Sobolev space $H^1(\Omega)$ is defined by

$$H^1(\Omega) := \left\{u \in L^2(\Omega) : \nabla u \in L^2(\Omega)^2\right\}.$$

The space $H_0^1(\Omega)$ contains all functions from $H^1(\Omega)$ with vanishing trace on the boundary $\partial\Omega$ of Ω. Let $\mathcal{T}$ be a triangulation of Ω consisting of quadrilaterals with possibly one-irregular hanging nodes. We assume that, for every $K \in \mathcal{T}$, there exists a reference mapping $F : \widehat{K} \to K$. Let $h := (h_K)_{K\in\mathcal{T}}$, $h_K := \mathrm{diam}(K)$ denote the mesh size vector and $p := (p_K)_{K\in\mathcal{T}}$, $p_K \in \mathbb{N}$ be the polynomial degree vector associated with triangulation $\mathcal{T}$. Further, we assume that $\mathcal{T}$ is (γ_h, γ_p)-regular [13–15]:

Definition 1 (*(γ_h, γ_p)-Regularity*). $\mathcal{T}$ is called (γ_h, γ_p)*-regular*, if and only if there exist constants $\gamma_h, \gamma_p > 0$ such that

$$\frac{h_{K_1}}{\gamma_h} \le h_{K_2} \le \gamma_h h_{K_1}$$

and

$$\frac{p_{K_1}}{\gamma_p} \le p_{K_2} \le \gamma_p p_{K_1}$$

for all $K_1, K_2 \in \mathcal{T}$ with $K_1 \cap K_2 \neq \emptyset$.

The finite-dimensional approximation space $V^p(\mathcal{T})$ is defined by

$$V^p(\mathcal{T}) := \left\{u \in H_0^1(\Omega) : u|_K \circ F_K \in Q_{p_K}\left(\widehat{K}\right) \text{ for all } K \in \mathcal{T}\right\},$$

where $Q_q\left(\widehat{K}\right)$ denotes the tensor-product polynomial space of degree $q \in \mathbb{N}$. An interior edge is the (nontrivial) intersection $e = K_1 \cap K_2$ of two elements $K_1, K_2 \in \mathcal{T}$ and we denote the collection of all interior edges by $\mathcal{E}(\mathcal{T})$.

Now, let $K \in \mathcal{T}$ be arbitrary. Then, we denote the set of all interior edges of cell K by $\mathcal{E}(\mathcal{T}; K)$. For $e \in \mathcal{E}(\mathcal{T}; K)$, we set $h_e := \mathrm{diam}(e)$ and $p_e := \max\left\{p_K, p_{K_*}\right\}$, where $K_* \in \mathcal{T}$ with $K \cap K_* = e$. Further, we define the patch ω_K around cell K by

$$\omega_K := \bigcup_{L\in\mathcal{T}} \{L : K \text{ and } L \text{ share a common edge}\}.$$

A slightly larger patch $\omega_{K,1}$ is defined by

$$\omega_{K,1} := \bigcup_{\substack{L\in\mathcal{T}\\ K\cap L\neq\emptyset}} L$$

and we can extend this definition iteratively by

$$\omega_{K,i+1} := \bigcup_{\substack{L\in\mathcal{T}\\ \omega_{K,i}\cap L\neq\emptyset}} L \tag{1}$$

for all $i \in \mathbb{N}$.