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Convexity-preserving approximation by univariate cubic splines

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ABSTRACT

A convexity-preserving approximation method is presented. Similar to the cubic spline interpolation, the given approximation function is univariate cubic spline with C^2 continuity. A very simple computing method is described. The approximation order and polynomial reproducing property of this convexity-preserving approximant are discussed. The computing method for interpolating a small quantity of data is given. Some numerical examples are given to show the effectivity.

1. Introduction

We often required to generate a smooth function that approximates a prescribed set of data. An approximant is often needed to preserve certain geometric shape properties of the data such as monotonicity or convexity. At the same time, the approximant is wanted to be C^1 or C^2 continuous.

There are many methods for constructing shape-preserving interpolation. Some methods are appropriate for monotonicity-preserving, see [1–4]. Some methods are appropriate for convexity-preserving, see [5–9]. There are some methods available for the construction of a C^1 shape preserving interpolant, see [10,6,3,11]. On the other hand, it is a more difficult task to construct a C^2 shape preserving interpolant, see [12,13,7,14,9].

Polynomial spline methods have a common feature in that no additional knots need to be supplied, see [15–17,13,2,18]. In contrast, the papers [19,4,20,14,9,11] discussed the methods by adding one or two additional knots on the subinterval so that the monotonicity or convexity of the data is preserved.

Rational interpolant can produce satisfied shape-preserving interpolation. For C^2 continuity, the solution of the consistency equations is concerned, see [5,21,6,12,1,22]. In the paper [7], the given interpolant is convexity-preserving and C^2 continuous without solving a global system of equations.

Recently, in the application to computer-aided design, some methods for constructing shape-preserving interpolation have been presented, see [23–27].

For shape-preserving approximation, some error estimates have been discussed in [28–34]. As we know, the approximation by B-splines is shape-preserving, see [35], but the approximation order is not satisfied. In [33], a simple convexity-preserving algorithm is given by B-splines.

Polynomial spline functions are the most widely used. They are simple and easy to compute. However, classical spline interpolation methods usually yield solutions exhibiting undesirable inflections or oscillations. The aim of this paper is to present an effective convexity-preserving approximation method by univariate cubic splines.

The remainder of this paper is organized as follows. In Section 2, the piecewise expression of the convexity-preserving approximation is described. A very simple computing method for constructing the cubic splines is given. Approximation

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properties of the presented cubic splines are discussed in Section 3. The computing method for interpolating a little inner data is presented in Section 4. Some numerical examples and conclusions are given in Sections 5 and 6 respectively.

2. Convexity-preserving cubic splines

Let $X = \{x_i : i = 1, 2, \ldots, n\}$ be a fixed partition of the interval $[a, b]$ by knots $a = x_1 < x_2 < \cdots < x_n = b$, and $D = \{f_i'' : i = 1, 2, \ldots, n\}$ be a set of n prescribed real numbers concerning the convexity of a given function $f(x)$ on X. If the data $f''(x_i)$ are given, we may set $f_i'' = f''(x_i),\ i = 1, 2, \ldots, n$. If only the data $f_i = f(x_i)$ are given, we may set

$$f_i'' = \frac{2}{h_{i-1} + h_i}(\Delta f_i - \Delta f_{i-1}) \tag{1}$$

for $i = 2, 3, \ldots, n-1$, and

$$f_1'' = f_2'', \qquad f_n'' = f_{n-1}'', \tag{2}$$

or natural boundary conditions

$$f_1'' = 0, \qquad f_n'' = 0, \tag{3}$$

where $h_i := x_{i+1} - x_i,\ \Delta f_i := [f(x_{i+1}) - f(x_i)]/h_i,\ i = 1, 2, \ldots, n-1$.

We will discuss the problem of convexity-preserving based on the given data f_i''. When all f_i'' have the same sign, based on the data $\{(x_i, f_i'') : i = 1, 2, \ldots, n\}$, we want to construct a concave approximation function (when $f_i'' \geq 0$) or a convex approximation function (when $f_i'' \leq 0$).

Definition. A convexity-preserving cubic spline on X is a real function S with properties:

(a) $S \in C^2[a, b]$, that is, S is twice continuously differentiable on $[a, b]$.
(b) S coincides on every subinterval $[x_i, x_{i+1}],\ i = 1, 2, \ldots, n-1$, with a polynomial of degree three.
(c) S satisfies $S''(x)f_i'' \geq 0$ on $[x_i, x_{i+1}]$ if $f_i'' f_{i+1}'' \geq 0$.

In the classical construction of a C^2 cubic interpolating spline, the second derivatives at knots are unknown and spline values $S(x_i) = f(x_i)$. Here the order is reversed: the spline values are considered as unknown and second derivatives $S''(x_i) = f_i''$. Thus, convexity-preserving cubic spline can be constructed explicitly. Based on the given second derivatives at knots, a cubic spline is piecewise convexity-preserving if the condition (c) holds.

Based on (b), the second derivative of the spline $S(x)$ coincides with a linear function in each interval $[x_i, x_{i+1}], i = 1, 2, \ldots, n-1$, and for (c), these linear functions can be described in terms of the moments f_i'' of S:

$$S''(x) = (1-t)f_i'' + tf_{i+1}'', \quad x \in [x_i, x_{i+1}], \tag{4}$$

where $i = 1, 2, \ldots, n-1$, $t = (x - x_i)/h_i$. This expression ensures that $S(x)$ is convexity-preserving. A data set may not be strictly concave or convex. If the signs of f_i'' and f_{i+1}'' are different by (1), then from (4) we can know that the curve $S(x)$ has exactly one inflection in interval $[x_i, x_{i+1}]$.

By integration, denote the free parameters by $S(x_i) = y_i$, $i = 1, 2, \ldots, n$, from (4) we have

$$\begin{aligned} S(x) = {} & (1-t)^3 y_i + 3(1-t)^2 t\left[\frac{1}{3}(2y_i + y_{i+1}) - \frac{h_i^2}{18}(2f_i'' + f_{i+1}'')\right] \\ & + 3(1-t)t^2\left[\frac{1}{3}(y_i + 2y_{i+1}) - \frac{h_i^2}{18}(f_i'' + 2f_{i+1}'')\right] + t^3 y_{i+1} \end{aligned} \tag{5}$$

for $x \in [x_i, x_{i+1}]$, and then

$$S'(x) = (1-t)^2\left[\Delta y_i - \frac{h_i}{6}(2f_i'' + f_{i+1}'')\right] + 2(1-t)t\left[\Delta y_i + \frac{h_i}{6}(f_i'' - f_{i+1}'')\right] + t^2\left[\Delta y_i + \frac{h_i}{6}(f_i'' + 2f_{i+1}'')\right], \tag{6}$$

where $\Delta y_i = (y_{i+1} - y_i)/h_i$. Obviously,

$$S'(x_i^+) = \Delta y_i - \frac{h_i}{6}(2f_i'' + f_{i+1}''), \qquad S'(x_{i+1}^-) = \Delta y_i + \frac{h_i}{6}(f_i'' + 2f_{i+1}'').$$

We need compute all y_i to determine the cubic spline (5). Based on (a), the continuity conditions $S'(x_i^-) = s'(x_i^+)$ yield the linear system

$$h_i y_{i-1} - (h_{i-1} + h_i)y_i + h_{i-1}y_{i+1} = h_{i-1}h_i b_i, \quad i = 2, 3, \ldots, n-1, \tag{7}$$

where

$$b_i = \frac{1}{6}[h_{i-1}f_{i-1}'' + 2(h_{i-1} + h_i)f_i'' + h_i f_{i+1}''].$$