Journal of Computational and Applied Mathematics 287 (2015) 207–219

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

An analytical coarse grid operator applied to a multiscale multigrid method

R.F. Sviercoski *, P. Popov, S. Margenov

Institute for Information and Communication Technologies, Bulgarian Academy of Sciences, Sofia, 1113, Bulgaria

ARTICLE INFO

Article history:
Received 31 March 2014
Received in revised form 25 February 2015

Keywords:
Homogenization and multigrid
Effective tensor
Cell-problem approximation
Analytical coarse operator
Analytical prolongation

ABSTRACT

Equations describing flow through heterogeneous porous media are often difficult to solve by direct numerical simulation given their expensive computational requirement. In this paper, numerical convergence is achieved for solving the flow equation with high contrast variable coefficients using two low computational methods that incorporate analytical approximations into the geometric multigrid. The methods are inheritably fast and easily implementable with coefficients describing general non-periodic media. The first method, named here as Analytical Coarse Operator (ACO), defines the operator at each coarse-scale of the V-cycle using an analytical approximation of the upscale tensor, it works in the same way as iterative homogenization. Its efficiency and reliability are measured by comparing with similar algorithms using arithmetic and harmonic averages of the coefficient, since they are widely used in the literature. The second method, called Analytical Prolongation Operator (APO), may be termed as aggressive coarsening, because it obtains the solution by skipping few levels in the V-cycle. It is an extension of the first approach where a multiscale prolongation operator is defined using an analytical approximation of the solution of the cell-problem from homogenization theory. The efficiency and reliability of this method are measured by comparing the number of V-cycles with the full numerical implementation. Various test cases demonstrated that convergence can be achieved using typical implementation procedures of the geometric multigrid method, therefore it highlights the low cost and easy implementation features of the methods. The examples include media having separable and non separable scales, periodic or not. In the case of non periodic medium, the problem of determining coarse grids that are representative element volume (REV) is also addressed. The medium from the SPE10 benchmark project is an application of the method for non separable scale.

1. Introduction

Multiscale problems for flows in porous media are usually encountered as part of research and development activities in a number of engineering, environmental and biomedical fields. Numerical algorithms for predicting flow behavior through systems, with complex hierarchical structures at each scale of interest, are still a major challenge in computational sciences, particularly if the requirement comes down to reliability and low computational cost methods. Even when the grid size can be considered small, these problems have high computational cost given the need to resolve all possible multiple scales. Standard applications can easily lead to numerical discretizations with very large number of degrees of freedom in the order

* Corresponding author.
E-mail address: rosviercoski@parallel.bas.bg (R.F. Sviercoski).

http://dx.doi.org/10.1016/j.cam.2015.03.001
0377-0427/

of 10^7–10^{10}. This, however, restricts everyday practical applications, as such simulations require advanced supercomputing resources and are impractical and/or impossible on commonly available computing equipment.

Given the complexity that these problems may present in various fields, a broad range of multiscale methods have been investigated in the last few decades. These include methods based on homogenization theory [1–3], generalized finite element methods [4,5], variational multiscale methods [6–9], multiscale domain decomposition methods [10], Multiscale Finite Element/Finite Volume methods (MsFEM/FV) [11,12], and domain decomposition methods together with energy minimization principles [13,14].

Classical methods based on homogenization theory are very often limited by strict assumptions on the media such as scale separation and periodicity, but these may not be always the case [15]. These methods can be described as one-way approach in which a cell solution in a Representative Element of Volume (REV) is used to compute an effective operator on the coarse level. Due to scale separation, the coarse equation is solved once to obtain a zeroth order (or averaged) approximation, then postprocessed using the cell solutions to obtain a fine-scale approximation to the true, fine-scale, solution.

In the absence of clear scale separation and highly heterogeneous coefficients, multiscale domain decomposition methods [10] and the closely related to MsFEM/FV have been used to obtain the fine-scale solution. Both types of methods assume a coarser level which cannot resolve the heterogeneous coefficient well, and a finer level which is at least one order of magnitude richer than the coarse one, thus being able to approximate well the fine-scale solution. They also depend crucially on solving local problems, which are then used to define the coarse scale operators as well as the prolongation to the fine level. In many aspects, these can be computationally demanding procedures.

More recently however, domain decomposition methods together with energy minimization principles [13,14] propose ways of reducing the computational cost for problems with high conductivity. The main idea is to use multiscale basis functions in the coarsest space that allow capturing the effects of inclusions that are isolated within the coarse grid blocks. In this case, a substantial reduction in computational cost can be achieved. However, in the high-conductivity, and when there are no isolation of inclusions, the dimension of the coarse space can still be very large, because smaller dimensional coarse spaces come at the expense of solving several lower dimensional problems in regions of high-anisotropy. These are examples of methods that can be robust with respect to the number of iterations, but can still have the limitations regarding the computational cost.

Other related approaches within the multilevel scope are obtained by using hierarchical iterative methods, such as the geometric multigrid together with homogenization [16–22]. These ideas are a natural generalization of multigrid that uses coarser grid solutions as initial guess solution to finer grid solutions. The homogenization fits as to better represent the coarser solutions when resolving problems with heterogeneous media. Moreover, the idea of using homogenization tools incorporated into multigrid schemes justify since in many interesting cases, the optimal performance of the multigrid alone is still a challenging problem. One case is when the heterogeneous media is described by step functions with high contrast coefficient. This lack of optimality may be due, for example, to the mesh anisotropy caused by elements having very large aspect ratio, typically appearing as a factor in the condition number of the stiffness matrix, which can easily generate a highly ill-conditioned problem [23]. An equally related problem is that the smallest eigenvalues do not correspond to very smooth eigenfunctions, making them difficult to represent on coarser grids [17].

Another limitation regards their generality, computational cost and implementation. For example, in [17,18,22], the algorithms are defined by prolongation/restriction operators for the special case of periodic oscillating media. For this particular case, closed forms showing convergence criteria as a function of the period ε, and fine grid discretization size h are available. Also, in the cited literature it has been highlighted how the convergence of the multigrid depends on the number of smoothings, generally associated with some measure of the heterogeneity. However, theoretical approaches together with practical implementation of such algorithms for general, non-periodic, media are unknown, and any attempt in that direction should include a way (or estimates) of dealing with possible boundary layers at each level.

The main goal of this paper is to propose two multiscale multigrid methods (FastMM) that have low computational cost, can be easily implemented and are more reliable than classical ones described in the literature. Numerical convergence results are presented for solving (1) with heterogeneous coefficient that are periodic or not periodic. Two complimentary methods are proposed, the ACO and APO. The first one uses an analytical approximation of the upscaled full tensor, $\mathbf{K}^\star$ from [15], to define coarse grid operators at each V-cycle level. It can be considered as an iterative homogenization procedure. The second method, the APO, is an aggressive coarsening procedure, obtained by using ACO and further defining a multiscale prolongation operator that captures fine-scale features, by incorporating analytical approximations for each periodic cell problem, from [24]. In both methods, the computational cost is comparable to the cost of the standard geometric multigrid. The underlying hypotheses in this work are that the effective coefficient, and the solution for the cell problems at the coarser levels, need not to be solved exactly, but can instead be replaced by suitable approximations.

The idea of using approximations of the upscaled coefficient is not new in the literature. The most popular are the arithmetic and harmonic average, or the combination of both [16,17] and, more recently [25] for the special case of bilinear FEM discretization and bilinear interpolation. Other example uses arithmetic average for the elasticity modules in the octree multigrid setting for bone microstructure analysis [26]. Because the approximations given by the arithmetic and harmonic averages are extreme bounds for the upscaled coefficient, each of them tend to perform better in one case (say, stiff inclusion) and not well in the other and vice-versa. Thus, closer approximations to the true upscaled value can lead to more reliable algorithms. By using this argument, some numerical methods find alternative ways of approximating numerically the upscaled coefficient, within the multilevel procedure. These numerical procedures that use only the zero-th