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A canonical form for the continuous piecewise polynomial functions

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ABSTRACT

We present in this paper a canonical form for the elements in the ring of continuous piecewise polynomial functions. This new representation is based on the use of a particular class of functions

$$\{C_i(P) : P \in \mathbb{Q}[x], i = 0, \ldots, \deg(P)\}$$

defined by

$$C_i(P)(x) = \begin{cases} 0 & \text{if } x \leq \alpha \\ P(x) & \text{if } x \geq \alpha \end{cases}$$

where α is the ith real root of the polynomial P. These functions will allow us to represent and manipulate easily every continuous piecewise polynomial function through the use of the corresponding canonical form.

It will be also shown how to produce a "rational" representation of each function $C_i(P)$ allowing its evaluation by performing only operations in $\mathbb{Q}$ and avoiding the use of any real algebraic number.

1. Introduction

The aim of this paper is to give a canonical representation for the elements in the ring of the continuous piecewise polynomial functions. While general piecewise polynomial functions are interesting in general, most applications of them to CAGD require the functions to be continuous. In fact, splines are, by definition, sufficiently smooth piecewise-defined polynomial functions. This, then, includes the special cases b-splines and NURBS. Since some important families of curves are continuous piecewise defined polynomials, it seems useful to have a specifically defined representation for them that can take advantage of such continuity.

In [1] von Mohrenschildt proposed a normal form, for piecewise polynomial functions, by means of the *step* functions

$$\text{step}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases}$$

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which are discontinuous. In [2], Chicurel-Uziel used characteristic functions of semilines to introduce a very natural form, with the same discontinuity issue. Furthermore, Carette [3] has worked on a canonical form for piecewise defined functions using range partitions.

We are interested, however, in representing canonically the continuous piecewise polynomial functions but based on a collection of continuous functions, which should prevent errors to grow out of control when evaluating on approximate numbers. Such a suitable set of continuous functions was introduced in [4,5]. They define, for every non-negative integer i, a mapping C_i from the set of polynomials on the set of continuous piecewise polynomial functions, such that

$$C_i(P)(x) = \begin{cases} 0 & \text{if } x \leq \alpha \\ P(x) & \text{if } x \geq \alpha \end{cases}$$

where α is the ith distinct real root of the polynomial P. If i is bigger than the number of real roots of P then $C_i(P)$ is defined as 0.

In [4,5] the $C_i(P)$ functions were used to study the Pierce–Birkhoff conjecture. This is a well-known and classical open problem in Real Algebraic Geometry asking if every continuous and piecewise polynomial function $h: \mathbb{R}^n \longrightarrow \mathbb{R}$ defined over $\mathbb{Q}$ can be represented by means of a sup–inf expression over a finite set of polynomials with rational coefficients. This conjecture has been proved in the affirmative sense only for $n = 1$ and $n = 2$ (see [4,5]) and remains still open for $n \geq 3$, while results in [6,7,11] lead to a proof in certain polyhedral domains.

In this paper we show that they provide a canonical representation which is easily computable from the piecewise expression of the functions. Moreover performing algebraic operations between canonical forms of continuous piecewise polynomial functions is simple and fast. In Section 2 we give the complete definition of the C_i functions along with some of their properties. Section 3 is devoted to the proof of existence and uniqueness of our canonical form. In Section 4 we show how to obtain easily the canonical form for the sum, product and composition of continuous piecewise polynomial functions. Section 5 shows how to produce a "rational" representation of each function $C_i(P)$ allowing its evaluation by performing only operations in $\mathbb{Q}$ and avoiding the use of any real algebraic number. Before the conclusions, Section 6 attacks some complexity aspects of the canonical form and the operations.

2. Preliminaries

Let us denote by $\mathcal{CP}(\mathbb{Q}[x])$ the set of continuous piecewise polynomial functions from $\mathbb{R}$ to $\mathbb{R}$ defined by polynomials with rational coefficients.

In order to represent canonically the continuous piecewise polynomial functions, we will use the set of mappings

$$C_i : \mathbb{Q}[x] \to \mathcal{CP}(\mathbb{Q}[x]),$$

$i \in \mathbb{N}$, presented in [4,5].

Definition 2.1. Let $P(x) \in \mathbb{Q}[x] \setminus \{0\}$, $\deg(P) = n$, $\{\alpha_1, \ldots, \alpha_r\}$ be the set of real roots of P, $\alpha_0 = -\infty$, $\alpha_k = +\infty$ for every $k > r$. Then, for every $i \in \mathbb{N} \cup \{0\}$, $x \in \mathbb{R}$,

$$C_i(P)(x) = \begin{cases} 0 & \text{if } x \leq \alpha_i \\ P(x) & \text{if } x \geq \alpha_i. \end{cases}$$

For completeness, we also define $C_i(P) = 0$ when $P = 0$.

The following result can be found in [4,5] as basis for the proof of Pierce–Birkhoff conjecture for the case one and two dimensional. It gives a natural representation of continuous piecewise polynomial functions in terms of the C_i functions.

Proposition 2.1. *Let ϕ be a continuous piecewise polynomial function*

$$\phi(x) = \begin{cases} Q_1(x) & \text{if } x \leq \alpha_1 \\ Q_2(x) & \text{if } \alpha_1 \leq x \leq \alpha_2 \\ \vdots & \\ Q_N(x) & \text{if } \alpha_{N-1} \leq x \end{cases}$$

with $Q_i \in \mathbb{Q}[x]$, $Q_i \neq Q_{i+1}$ for all i, and $\alpha_j \in \mathbb{R}$. Then ϕ can be written in the following way:

$$\phi(x) = Q_1(x) + \sum_{i=1}^{N-1} C_{s(i)}(\Delta_i)(x)$$

where $\Delta_i = Q_{i+1} - Q_i$ and $s(i)$ is the position index of α_i as a root of Δ_i.

Proof. For every $i \in \{1, \ldots, N-1\}$ we define the polynomial in $\mathbb{Q}[x]$ given by:

$$\Delta_i(x) = Q_{i+1}(x) - Q_i(x).$$