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Direct solution of a type of constrained fractional variational problems via an adaptive pseudospectral method

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ABSTRACT

This paper presents an adaptive Legendre–Gauss pseudospectral method for solving a type of constrained fractional variational problems (FVPs). The fractional derivative is defined in the Caputo sense. In the presented method, by dividing the domain of the problem into a uniform mesh the given FVP reduces to a nonlinear mathematical programming problem, and there is no need to solve the complicated fractional Euler–Lagrange equations. The method developed in this paper adjusts both the mesh spacing and the number of collocation points on each subinterval in order to improve the accuracy. The method is easy to implement and yields very accurate results. Some error estimates and convergence properties of the method are discussed. Numerical examples are included to confirm the efficiency and convergence of the proposed method.

1. Introduction

In recent decades, fractional calculus, a classical branch of mathematical analysis that studies non-integer order derivatives and integrals, has found numerous applications in several different disciplines [1–4]. In this framework, the fractional calculus of variations is a research area under strong current development. A fractional variational problem (FVP) consists in finding the extremum of a functional that depends on fractional derivatives and/or integrals subject to some boundary conditions and possibly some extra constraints. Different definitions of fractional derivatives and/or integrals can be considered in FVPs, however, the most important types are Riemann–Liouville and Caputo definitions. The study of FVPs was introduced by Riewe [5]. He found out that traditional Lagrangian and Hamiltonian mechanics cannot be used with nonconservative forces such as friction. In fact, the systems in Nature always contain internal damping and are subject to some external forces that do not store energy and which are not equivalent to the gradient of a potential i.e., they are not conservative. For nonconservative dynamical systems the energy conservation law is broken and, as a consequence, the standard Hamiltonian formalism is no longer valid for describing the behavior of the system. Riewe [5] showed that a Lagrangian involving fractional time derivatives leads to an equation of motion with nonconservative forces. It is a remarkable result since frictional and nonconservative forces are beyond the usual macroscopic variational treatment, and consequently, beyond the most advanced methods of classical mechanics. Riewe also showed that fractional formalism can be used when treating

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dissipative problems. By inserting fractional derivatives into the variational integrals he obtained the respective fractional Euler–Lagrange equation, combining both conservative and nonconservative cases. The interested reader can refer to the comprehensive recent book [6], and for models and numerical methods we refer to [7].

There are two major approaches in the classical theory of calculus of variations. In one hand, using Euler–Lagrange necessary optimality conditions, we can reduce a variational problem to the study of a differential equation. Hereafter, one can use either analytical or numerical methods to solve the differential equation and reach the solution of the original problem (see, e.g., [8]). This approach is referred as indirect methods in the literature. On the other hand, we can tackle the functional itself, directly. Direct methods are used to find the extremum of a functional in two ways: Euler's finite differences and spectral methods including Ritz, Galerkin and pseudospectral schemes [9]. In spectral methods, we either approximate admissible functions by all possible linear combinations $y_N(t) = \sum_{i=1}^{N} a_i\phi_i(t)$, with constant coefficients a_i and a set of known base functions ϕ_i, or we approximate the admissible functions with such combinations. By finite differences, however, we consider the admissible functions not on the class of arbitrary curves, but only on polygonal curves made upon a given grid on the time horizon. Using an appropriate discrete approximation of the Lagrangian, and substituting the integral with a sum, we can transform the main problem to the optimization of a function of several parameters.

Indirect methods for FVPs have a vast background in the literature and can be considered a well studied subject: see [10–23] and references therein that study different variants of the problem and discuss a bunch of possibilities in the presence of fractional terms, Euler–Lagrange equations and boundary conditions. There also exist some indirect numerical methods for solving FVPs. For instance, fractional variational integrators in [24–26] are developed and applied for FVPs, and in [27] a numerical scheme is proposed for solving a class of parametric FVPs. However, since in indirect methods the necessary optimality conditions combine the fractional differential operators, it is hard to find the exact solution of the problem. Even if we try to solve them approximately, we will face with complexity in computations.

In recent years, several direct methods for FVPs have been proposed in the literature. A brief introduction of using finite differences has been made in [5]. Leitmann's direct method for fractional optimization problems has been proposed in [28]. Authors of [29] discuss the Ritz direct method for a class of FVPs with multiple dependent variables, multi order fractional derivatives and a group of boundary conditions. A fractional finite element formulation is introduced in [30] for solving quadratic form FVPs. In [31] the idea of finite differences direct method is generalized for FVPs. Authors of [32–34] derived three approximations from continuous expansions of Riemann–Liouville fractional derivatives and integrals into series involving integer order derivatives. Then, they illustrated the usefulness of their obtained results, in solving different problems such as fractional problems of the calculus of variations and fractional order optimal control problems. The fact that the first variation of a variational functional must vanish along an extremizer is generalized in [35] to solve FVPs.

Another class of direct methods are direct pseudospectral methods. During the past few years, pseudospectral methods have been successfully used to solve a wide variety of variational problems and optimal control problems [36–38], owing to its high order of accuracy. Pseudospectral methods can be interpreted as direct transcription methods for discretizing a continuous variational problem into a nonlinear programming problem by parameterizing the rate variables using polynomials and nodes obtained from a Gaussian quadrature, see, e.g., [37,39]. Pseudospectral methods arose from spectral methods which were traditionally used to solve fluid dynamics problems [40,41]. They can often achieve higher accuracy than finite difference scheme or finite element method [42]. The basis functions in pseudospectral methods are typically Chebyshev or Legendre polynomials. The key point in pseudospectral methods is that they avoid the poor behavior of the classical polynomial interpolation methods by removing the restriction of equally spaced interpolation points. The approaches based on pseudospectral methods can be divided into *global* and *adaptive* pseudospectral methods. Global pseudospectral methods use global polynomials together with Gaussian quadrature collocation points which are known to provide accurate approximations that converge exponentially for problems whose solutions are smooth and well-behaved over the whole domain of interest [38]. However, global pseudospectral methods also suffer from some drawbacks. For instance, they do not provide a satisfactory approximation for nonsmooth problems. On the other hand, adaptive pseudospectral methods increase the utility of pseudospectral methods while attempting to maintain as close to exponential convergence as possible. They allow the number of subintervals, subinterval widths, and polynomial degrees to vary throughout the time interval of interest. Recently, Darby et al. [43] and Maleki et al. [44] presented new adaptive pseudospectral methods for solving non-delay and delay optimal control problems of integer order, respectively.

Also, Maleki et al. [45] proposed an adaptive pseudospectral method for solving a class of multi-term fractional boundary value problems. Adaptive pseudospectral methods, however, to the best of our knowledge, are not well studied for solving nonlinear FVPs.

In this paper, we generalize the idea presented in [45] for direct solution of a type of constrained FVPs. We consider a minimization problem with a Lagrangian that depends on the Caputo fractional derivative and involves inequality constraints. This method is based on piecewise interpolation using the shifted Legendre–Gauss collocation points. A pseudospectral scheme is introduced for approximating the fractional derivatives of order $0 < \alpha < 1$ of the rate variables at the shifted Legendre–Gauss points. We investigate some error estimates and convergence properties. One of the advantages of the proposed method is the good representation of smooth and especially piecewise smooth functions.

The outline of this paper is as follows: in Section 2, we present some preliminaries needed for our subsequent developments. In Section 3 the statement of the problem is given. Legendre–Gauss piecewise interpolation and its fractional derivatives are explained in Section 4, and Section 5 is devoted to the proposed direct method. Some error estimates and the convergence of the method are investigated in Section 6. In Section 7, we report our numerical findings and demonstrate