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Mixed finite element methods for two-body contact problems

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ABSTRACT

This paper presents mixed finite element methods of higher-order for two-body contact problems of linear elasticity. The discretization is based on a mixed variational formulation proposed by Haslinger et al. which is extended to higher-order finite elements. The main focus is on the convergence of the scheme and on a priori estimates for the *h*- and the *p*-method. For this purpose, a discrete inf–sup condition is proven which guarantees the stability of the mixed method. Numerical results confirm the theoretical findings.
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1. Introduction

The aim of this paper is to derive mixed finite element methods of higher-order for two-body contact problems in linear elasticity. The discretization approach is based on mixed finite elements for contact problems introduced by Haslinger et al. in [1–3]. This approach was originally developed for low-order finite elements. In this paper, we extend it to higher-order discretizations and to two-body contact problems. The approach relies on a saddle point formulation. The introduced Lagrange multiplier is defined on the surface of one of the bodies in contact and enforces the geometrical contact condition via a sign condition.

To guarantee the uniqueness of the solution of the mixed scheme and to show its convergence one has to provide a uniform discrete inf–sup condition which balances the discretization spaces of the primal variable and of the Lagrange multiplier. It is an essential assumption to show the convergence of the mixed scheme without regularity assumptions, to derive a priori estimates and to determine convergence rates based on these estimates.

In this work, the higher-order discretization of the primal variable is given via a conforming ansatz using tensor product polynomials. The discretization space of the discrete Lagrange multiplier is also based on such tensor products. To include the sign condition, we enforce the discrete Lagrange multiplier to be positive only in Gauss quadrature points leading to a non-conforming discretization. This approach was already suggested in [4] for frictional contact problems. We show the convergence of the mixed scheme and discuss some arguments as proposed by Haslinger et al. and Lhalouani et al., cf. [1,2,5–7] to determine convergence rates for low-order discretizations of the Lagrange multiplier. The main result is the derivation of convergence rates with respect to higher-order discretizations in both variables. The essential ingredient is to intensively utilize the discretization of the Lagrange multiplier via its definition in Gauss points. This enables to apply higher-order interpolations as introduced in [8] as well as quadrature rules for the exact integration of polynomials.

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This work also deals with the verification of a uniform discrete inf–sup condition. For low-order finite elements and one-body contact problems, the discrete inf–sup condition is proven in [1,2]. An essential assumption of the proof is that the discretization of the Lagrange multiplier is defined on boundary meshes with a different mesh size than that of the primal variable. We show that in the higher-order approach, this assumption can, in principle, be avoided using different polynomial degrees. In the proof of the discrete inf–sup condition, we use approximation results of the p-method of finite elements and some inverse estimates for higher-order polynomials, cf. [9,10]. In particular, we adapt the proof of the discrete inf–sup condition for frictional one-body contact problems as described in [11].

Higher-order discretization schemes for contact problems are rarely studied in literature, especially for mixed variational formulations. We refer the reader to [12,13] for finite element discretizations based on primal, non-mixed formulations, to [4] for mixed methods using a mortar approach and to [14] for boundary element methods. Mixed methods with quadratic finite elements are described in [15–17].

The paper is organized as follows: In Section 2, the two-body contact problem and its mixed variational formulation are introduced. The convergence of the mixed scheme and general a priori estimates are discussed in Section 3. In Section 4 the discretization of higher-order is presented, its convergence is proven and convergence rates are determined. A uniform discrete inf–sup condition is proven in Section 5. Finally, numerical results confirming the theoretical findings are discussed in Section 6.

2. Two-body contact problem and its mixed variational formulation

We consider the deformation of two bodies being in contact. They are represented by the domains $\Omega^l \subset \mathbb{R}^k$, $k \in \{2, 3\}$, $l \in \{1, 2\}$, with sufficiently smooth boundaries $\Gamma^l := \partial\Omega^l$ and are clamped at some boundary parts which are represented by the closed sets $\Gamma_D^l \subset \Gamma^l$ with positive measure. The boundary parts of the bodies where the bodies possibly get in contact are described by open sets Γ_C^l where we assume $\overline{\Gamma}_C^l \subsetneq \Gamma^l \setminus \Gamma_D^l$ and $\Gamma_N^l := \Gamma^l \setminus (\Gamma_D^l \cup \overline{\Gamma}_C^l)$. Volume and surface forces act on the bodies. They are described by functions $f^l \in L^2(\Omega^l; \mathbb{R}^k)$ and $q^l \in L^2(\Gamma_N^l; \mathbb{R}^k)$. The resulting deformation is described by displacement fields $v^l \in H^1(\Omega^l; \mathbb{R}^k)$ with the linearized strain tensor $\varepsilon(v) := \frac{1}{2}(\nabla v + (\nabla v))^\top$. The stress tensor describing the linear-elastic material law is defined as $\sigma^l(v)_{ij} := \mathcal{C}^l_{ijkl}\varepsilon(v)_{kl}$, where $\mathcal{C}^l_{ijk\ell} \in L^\infty(\Omega)$ with $\mathcal{C}^l_{ijk\ell} = \mathcal{C}^l_{ji\ell k} = \mathcal{C}^l_{k\ell ij}$ and $\mathcal{C}^l_{ijk\ell}\tau_{ij}\tau_{k\ell} \geq \kappa\tau_{ij}^2$ for all $\tau \in L^2(\Omega; \mathbb{R}^{k\times k})$ with $\tau_{ij} = \tau_{ji}$ and a constant $\kappa > 0$. We set $H_D^1(\Omega^l) := \{v \in H^1(\Omega^l; \mathbb{R}^k) \mid \gamma^l_{|\Gamma_D^l}(v_i) = 0,\ i = 1, \dots, k\}$ for the trace operator $\gamma^l \in L(H^1(\Omega^l), L^2(\Gamma^l))$ and define $(\sigma_n^l(v^l))_i := \sigma_{ij}^l(v^l)n_j^l$, $v_n^l := v_i^l n_i^l$, $\sigma_{nn}^l(v^l) := \sigma_{ij}^l(v^l)n_i^l n_j^l$, $\sigma_{nt}^l(v^l) := \sigma_n^l(v^l) - \sigma_{nn}^l(v^l)n$ with outer normal n^l of Γ^l. For a bijective, sufficiently smooth mapping $\Phi : \Gamma_C^1 \to \Gamma_C^2$ and $x \in \Gamma_C^1$, we define

$$\tilde{n}(x) := \begin{cases} \dfrac{\Phi(x) - x}{|\Phi(x) - x|}, & x \neq \Phi(x), \\ n^1(x) = -n^2(x), & x = \Phi(x) \end{cases}$$

and the gap function $g(x) := |x - \Phi(x)|$. Furthermore, we set $[v^1, v^2]_{\tilde{n}}(x) := v_i^1(x)\tilde{n}_i - v_i^2(\Phi(x))_i\tilde{n}_i$ for functions v^1 and v^2 on Γ_C^1 and Γ_C^2, respectively. The two-body contact problem is thus to find displacement fields u^1 and u^2 such that

$$\begin{aligned} &-\mathrm{div}\,\sigma^l(u^l) = f^l \quad \text{in } \Omega^l, \\ &u^l = 0 \quad \text{on } \Gamma_D^l, \\ &\sigma_n^l(u^l) = q^l \quad \text{on } \Gamma_N^l, \\ &\sigma_{nt}^l(u^l) = 0 \quad \text{on } \Gamma_C^l, \\ &[u^1, u^2]_{\tilde{n}} \leq g, \qquad \sigma^1_{\tilde{n}\tilde{n}}(u^1) \leq 0, \qquad \sigma^1_{\tilde{n}\tilde{n}}(u^1)([u^1, u^2]_{\tilde{n}} - g) = 0 \quad \text{on } \Gamma_C^1. \end{aligned}$$

In this paper, the following notational conventions are used. The space $H^{-1/2}(\Gamma_C^1)$ denotes the topological dual space of $H^{1/2}(\Gamma_C^1)$ with norms $\|\cdot\|_{-1/2,\Gamma_C^1}$ and $\|\cdot\|_{1/2,\Gamma_C^1}$, respectively. Let $(\cdot,\cdot)_{0,\omega}$ and $(\cdot,\cdot)_{0,\Gamma'}$ be the usual L^2-scalar products on $\omega \subset \mathbb{R}^k$ and $\Gamma' \subset \Gamma_C^1$, respectively. For $v \in H_D^1(\Omega^l)$ and $w \in L^2(\Gamma')$, we define $\|v\|^2_{0,\Omega^l} := (v_i, v_i)_{0,\Omega^l}$ and $\|w\|^2_{0,\Gamma'} := (w, w)_{0,\Gamma'}$. Furthermore, the usual H^1-norm on $H_D^1(\Omega^l)$ is denoted by $\|\cdot\|_{1,\Omega^l}$. We define $\gamma_N^l \in L(H_D^1(\Omega), L^2(\Gamma_N^l, \mathbb{R}^k))$ as $\gamma_N^l(v)_i := \gamma^l_{|\Gamma_N^l}(v_i)$ and $\mathcal{H}_D := H_D^1(\Omega^1) \times H_D^1(\Omega^2)$, which is a Hilbert space with the norm $\|v\|_1^2 := \sum_{l=1,2}\|v^l\|^2_{1,\Omega^l}$ for $v \in \mathcal{H}_D$. We set $\gamma_{C\tilde{n}} \in L(\mathcal{H}_D, H^{1/2}(\Gamma_C^1))$ as $\gamma_{C\tilde{n}}(v) := [\gamma_C^1(v^1), \gamma_C^2(v^2)]_{\tilde{n}}$ which is surjective due to the assumptions on Γ_C^1, cf. [18]. Finally, we introduce some interpolation spaces $H^{1+\theta}(\Omega^l)$ and $H^{-1/2+\theta}(\Gamma_C^1)$ for $\theta > 0$ which are defined via $H^{1+\theta}(\Omega^l) := [H^1(\Omega^l), H^2(\Omega^l)]_{\theta,2}$ and $H^{-1/2+\theta}(\Gamma_C^1) := [H^{-1/2}(\Gamma_C^1), H^{1/2}(\Gamma_C^1)]_{\theta,2}$ with norms $\|\cdot\|_{1+\theta,\Omega^l}$ and $\|\cdot\|_{-1/2+\theta,\Gamma_C^1}$, respectively, cf. [19,20].

It is well-known that the solution of the two-body contact problem $u \in \mathcal{H}_D$ is also a solution $u \in K := \{v \in \mathcal{H}_D \mid \gamma_{C\tilde{n}}(v) \leq g\}$ of the variational inequality

$$a(u, v - u) \geq \ell(v - u)$$

for all $v \in K$, where $a(u, v) := \sum_{l=1,2}(\sigma_{ij}(u^l), \varepsilon_{ij}(v^l))_0$ and $\ell(v) := \sum_{l=1,2}\left((f_i^l, v_i^l)_0 + (q_i^l, \gamma_N^l(v^l)_i)_{0,\Gamma_N^l}\right)$. The inequality above is fulfilled if and only if u is a minimizer of the functional $E(v) := \frac{1}{2}a(v, v) - \ell(v)$ in K. Due to Cauchy's and Korn's

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