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A connection between birational automorphisms of the plane and linear systems of curves[☆]

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ABSTRACT

In this paper, we prove that there exists a one-to-one correspondence between birational automorphisms of the plane and pairs of pencils of curves intersecting in a unique point. As a consequence, we show how to construct birational automorphisms of the plane of a certain degree d (fixed in advance) from some curves generating two linear systems of curves of degrees d and $\widetilde{d}$, where $\widetilde{d} = d - 2$ for $d > 2$, and $\widetilde{d} = 1$ otherwise. In addition, we also get the inverse of the birational automorphism constructed, and we show that its degree is obtained from the degree of the linear system of curves. As a special case, we show how these results can be stated to polynomial birational automorphisms of the plane.

1. Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. The group of birational automorphisms of the projective space $\mathbb{P}^n$ over $\mathbb{K}$ is called the group of Cremona transformations of $\mathbb{P}^n$ or the Cremona group. There is an extensive classical literature about this group (see for instance, [1–11]).

It is well known that birational automorphisms of the parameter space $\mathbb{P}^1$ are the Möbius transformations. In the plane, by Noether's Theorem, if $\mathbb{K}$ is an algebraically closed field, each Cremona transformation can be expressed as a composition of quadratic transformations. The simplest examples of Cremona transformations may be given as linear-fractional transformations

$$\mathcal{P}(t_1, t_2) = \left(\frac{a_1t_1 + b_1t_2 + c_1}{a_2t_1 + b_2t_2 + c_2}, \frac{a_3t_1 + b_3t_2 + c_3}{a_4t_1 + b_4t_2 + c_4}\right).$$

Cremona transformations of $\mathbb{P}^1$ and $\mathbb{P}^2$ have been extensively used by algebraic geometers and in particular, they play an important role in the frame of the algebraic manipulations of curves and surfaces. For instance, they are effective in the reduction of singularities of curves to points with distinct tangents (see [12]). That is, a plane algebraic curve can be transformed by a Cremona transformation into a plane algebraic curve with ordinary multiple points. For surfaces, it is also shown that every algebraic surface can be transformed by a Cremona transformation into a surface having only ordinary multiple curves (for further details see [13]). In [14,15], the reduction of linear systems of plane curves by Cremona transformations is considered. There, one attempts to do a reduction by quadratic Cremona transformation which gives in turn a proof of the classical result that the Cremona transformations are generated by the quadratic ones (see [16]).

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On the other side, it is well known that, in the case of algebraic curves, Möbius transformations preserve the degree of a given curve's parametrization. Hence the parametric degree (that is, the degree of a parametrization) is the same for all proper parametrizations. In the case of surfaces, the parametric degree is not preserved by Cremona transformations. However, in general, one can find a Cremona transformation that reduces the degree of a surface's parametrization (see [17]).

For these reasons, the problem of easily constructing Cremona transformations, by controlling its degree, is very important. In fact, several authors have dealt with this question recently (see e.g. [18,19]). Thus, given $d \in \mathbb{N}$, in this paper we are interested in constructing a birational automorphism of the plane $\mathcal{S}$ such that $\deg(\mathcal{S}) = d$. For this purpose, we construct a pencil of curves $\mathcal{V}_1$ of degree d (where singularities and simple points are known), and using *Algorithm for Pencil Parametrization* (see Section 2) we determine a linear subsystem $\mathcal{V}_2$ of dimension 1 of the system of adjoint curves to $\mathcal{V}_1$. From the curves generating $\mathcal{V}_1$ and $\mathcal{V}_2$, we obtain $\mathcal{S}$. From this construction, we get that the inverse of $\mathcal{S}$ is the unique non-constant intersection point of $\mathcal{V}_1$ and $\mathcal{V}_2$, and its degree is obtained from the degree of $\mathcal{V}_1$ and $\mathcal{V}_2$.

Reciprocally, we show how a given birational automorphism is related with a pair of pencils that intersect in a unique point. More precisely, we are given a birational automorphism $\mathcal{P}$, and we construct a pair of pencils $(\mathcal{V}_1, \mathcal{V}_2)$, from which we obtain a birational automorphism $\mathcal{S}$. We show that "up to composition with a polynomial De Jonquières transformation", $\mathcal{P}$ is *equivalent* to $\mathcal{S}$ (that is, $\mathcal{P} = \mathcal{J} \circ \mathcal{S}$, where $\mathcal{J}$ is a De Jonquières transformation). Therefore, in this paper, we prove that there exists a one-to-one correspondence between birational automorphisms of the plane and pairs of pencils $(\mathcal{V}_1, \mathcal{V}_2)$ intersecting in a unique point.

Finally, we also show how these results can be stated similarly for the case of birational automorphisms of the plane that are polynomial and thus, in particular, we construct polynomial birational automorphisms of the plane of a desired degree d. We remark that polynomial automorphisms have an additional interest in practical applications, since the non-existence of denominators avoids the possible unstable behavior, when the parameters take values close to the points of the curves defined by the denominators (see e.g. [20]).

For higher dimension there has also been a lot of research on the subject [21–28], though the results obtained remain sporadic and, in general, there are no substantial advances with respect to the pioneering works in the knowledge either about the structure of arbitrary Cremona transformations themselves or about the structure of the group of Cremona transformations, even for $n = 3$. The results presented here attempt to open several ways that can be used to provide significant results concerning Cremona transformations for $n \geq 3$ (see Section 4).

The structure of the paper is as follows: in Section 2, we provide some preliminaries and previous results. For this purpose, two subsections are considered: in Section 2.1, we deal with the classical problem of parametrizing a plane curve over a subfield k of and algebraically closed field K of characteristic zero, and the definition of k-rational curve is introduced (see Definition 1). In Section 2.2, we specialize Section 2.1 to the case where the input curve is a pencil of curves, $k = \mathbb{K}(t)$ and $K = \overline{\mathbb{K}(t)}$ being $\mathbb{K}$ an algebraically closed field of characteristic zero. Section 3 is devoted to show a one-to-one correspondence between birational automorphisms of the plane and pairs of pencils intersecting in a unique point. For this purpose, three subsections are considered: in Section 3.1, we show how birational automorphisms of the plane of a desired degree d can be constructed from the curves generating two 1-dimensional systems of curves of degrees d and $\widetilde{d}$, where $\widetilde{d} = d-2$ for $d > 2$, and $\widetilde{d} = 1$ otherwise (see Theorem 1 and statement 2 in Corollary 2). In Section 3.2, we show how a given birational automorphism is related with a pair of pencils. More precisely, we are given a birational automorphism $\mathcal{P}$, and we construct a pair of pencils $(\mathcal{V}_1, \mathcal{V}_2)$ intersecting in a unique point, from where we obtain a birational automorphism $\mathcal{S}$. We show that "up to composition with a polynomial De Jonquières transformation", $\mathcal{P}$ is *equivalent* to $\mathcal{S}$ (see Theorem 2). In Section 3.3, we show how these results can be stated similarly to the case of birational polynomial automorphisms (see Proposition 1 and Theorem 3). We finish with a section with conclusions and open questions (Section 4).

2. Preliminaries and previous results

In this section, we introduce the notation and some previous algorithmic methods and results that will be used throughout the paper. In Section 2.1, we recall basic facts about parametrizations of rational curves. In particular, we consider K an algebraically closed field of characteristic zero and we treat briefly the classical problem of parametrizing an algebraic plane curve $\mathcal{C}^* \subset \mathbb{P}^2(K)$ over a subfield $k \subseteq K$. Afterwards, in Section 2.2, we specialize Section 2.1 to the case where $\mathcal{C}^*$ is a pencil of curves, $k = \mathbb{K}(t)$ and $K = \overline{\mathbb{K}(t)}$ (the algebraic closure of the field $\mathbb{K}(t)$) being $\mathbb{K}$ an algebraically closed field of characteristic zero.

2.1. Parametrization of a rational curve

Let K be an algebraically closed field of characteristic zero, and let $k \subseteq K$ a subfield. We denote by $\mathbb{A}^2(K)$, the affine plane embedded into the projective plane $\mathbb{P}^2(K)$ by identifying the point $(a, b) \in \mathbb{A}^2(K)$ with the point $(a : b : 1) \in \mathbb{P}^2(K)$. Hence, given an affine algebraic plane curve $\mathcal{C}$ over K, we denote by $\mathcal{C}^*$ the corresponding projective algebraic curve, i.e. the projective closure of $\mathcal{C}$ in $\mathbb{P}^2(K)$.

If the affine curve $\mathcal{C}$ is defined by a polynomial $f(x) \in K[x]$, $x = (x_1, x_2)$, the corresponding projective curve $\mathcal{C}^*$ is defined by the homogenization $F(X) \in K[X]$, $X = (x_1 : x_2 : x_3)$, of $f(x)$. Thus, $\mathcal{C}^* = \{(a : b : c) \in \mathbb{P}^2(K) \mid F(a, b, c) = 0\}$, and every