Journal of Computational and Applied Mathematics 283 (2015) 163–181

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Finite Elements with mesh refinement for wave equations in polygons

Fabian L. Müller *, Christoph Schwab

Seminar for Applied Mathematics, Rämistrasse 101, 8092 Zurich, Switzerland

ARTICLE INFO

Article history:
Received 3 April 2013
Received in revised form 21 October 2014

Keywords:
High order Finite Elements
Wave equation
Regularity
Method of lines
Local mesh refinement
Newest vertex bisection

ABSTRACT

Error estimates for the space semi-discrete approximation of solutions of the Wave equation in polygons $G \subset \mathbb{R}^2$ are presented. Based on corner asymptotics of the solution, it is shown that for continuous, simplicial Lagrangian Finite Elements of polynomial degree $p \geq 1$ with either suitably graded mesh refinement or with bisection tree mesh refinement towards the corners of G, the maximal rate of convergence $O(N^{-p/2})$ which is afforded by the Lagrangian Finite Element approximations on quasiuniform meshes for smooth solutions is restored. Dirichlet, Neumann and mixed boundary conditions are considered. Numerical experiments which confirm the theoretical results are presented. Generalizations to nonhomogeneous coefficients and elasticity and electromagnetics are indicated.

1. Introduction

The regularity of elliptic equations in polygonal domains has been studied for several decades, starting with the work by Kondrat'ev [1] and Maz'ya and Plamenevskiĭ [2]. We refer to Maz'ya and Rossmann [3] for a recent account of these results, also in polyhedral domains in $\mathbb{R}^3$ and a comprehensive list of references.

It is well-known that regularity results in scales of Sobolev spaces with weights allow to recover optimal convergence rates for Finite Element Methods (FEM) with local mesh refinement in the vicinity of corners; we refer to Raugel [4] and Babuška et al. [5,6], and Băcuţă et al. [7] for so-called *graded meshes*, and, more recently, to [8] and the references there for simplicial meshes with bisection tree refinements produced by Adaptive Finite Element Methods (AFEMs).

For evolution problems, in particular for the linear, second order Wave equation, similar results do not seem to be available. However, corner singularities are known to play a crucial role in the scattering and diffraction of waves. In recent years, results on the regularity of the pure Dirichlet and Neumann problems of solutions of the Wave equation in polygonal and in certain polyhedral domains have been proved by Plamenevskiĭ et al. in [9–11] for the scalar, acoustic Wave equation, and in [12,13] for a general class of second order, linear hyperbolic systems. Their results imply that at a fixed time t, $u(\cdot, t)$ belongs to a class of function spaces $H^{p+1,2}_{\delta}$ which appeared already in the study of elliptic equations. Moreover, in these papers explicit formulae for the asymptotics of $u(x, t)$ in the vicinity of corners of the polygon G were obtained. Therefore, in principle, approximation results for $H^{p+1,2}_{\delta}$ on several families of locally refined meshes as, for example, in [6], as well as a mesh refinement algorithm presented in [8], may now be applied to the solution of the Wave equation. The main result of the present paper is that the space semi-discrete ("method of lines") type discretization of the Wave equation yields optimal convergence rates for solutions with singular asymptotic behaviour in the vicinity of the corners, which are known

* Corresponding author.
E-mail addresses: fabian.mueller@sam.math.ethz.ch (F. Müller), christoph.schwab@sam.math.ethz.ch (Ch. Schwab).

http://dx.doi.org/10.1016/j.cam.2015.01.002

to typically occur in solutions of the linear, second order Wave equation. We hasten to add that our approximation results are also applicable to singularities which arise in propagation of elastic and electromagnetic waves in polygonal domains.

The outline of the present paper is as follows. We start with an introduction to the used notations, and the formulation of the scalar Wave equation with Dirichlet and Neumann conditions in Section 2. Section 3 contains a review of the regularity theory for the scalar Wave equation, starting from the definitions of weighted Sobolev spaces. In Section 4, we study the FEM-approximation of singular functions, and recall two classes of meshes which yield optimal convergence rates in the presence of corner singularities. These results are applied in Section 5 with the decomposition theorem to obtain optimal convergence rates for the space semi-discrete Finite Element approximation of the Wave equation in polygonal domains. Finally, in Section 6, we present results of numerical experiments, performed with a very small time-step to approximately "cancel" the influence of the time-stepping error.

2. Problem formulation

On an open, bounded polygonal domain $G \subseteq \mathbb{R}^2$ and for $0 < T_{\max} < \infty$, with boundary $\partial G = \Gamma_D \cup \Gamma_N$ which consists of a finite number of straight segments Γ_i which are partitioned into Dirichlet and Neumann segments, we consider the initial–boundary value problem for the scalar Wave equation with Dirichlet or Neumann boundary conditions, i.e. we wish to find solutions $u(x,t),\ (x,t) \in Q_{\text{fin}} := G \times (0, T_{\max})$ such that

$$\begin{aligned} u_{tt} - \Delta u &= f && \text{in } Q_{\text{fin}}, \\ u(\cdot,0) &= u_0 && \text{in } G, \\ u_t(\cdot,0) &= v_0 && \text{in } G, \\ u(\cdot,t) &= 0 && \text{on } \Gamma_D \times (0,T_{\max}), \\ \frac{\partial u}{\partial n} &= 0 && \text{on } \Gamma_N \times (0,T_{\max}). \end{aligned} \tag{1}$$

We denote by $H^s(G)$ the usual Sobolev spaces on G, and by $H_0^1(G)$ the subspace of $H^1(G)$ built by functions with vanishing trace. Moreover, given a Hilbert space H, we denote by $H^s(0,T_{\max};H)$ the H^s-Bochner space of functions from $[0,T_{\max}]$ to H. We introduce the space V defined as the completion of $\{v \in C^\infty(G) : \ v|_{\Gamma_D} \equiv 0\}$ with respect to the H^1-norm. We set

$$V = \begin{cases} H_0^1(G) & \text{if } \Gamma_N = \emptyset, \\ H^1(G) & \text{if } \Gamma_D = \emptyset. \end{cases}$$

We will also denote by $(\cdot,\cdot)$ the $L^2(G)$ inner product, extended to the pair of spaces $V \times V^*$ with duality taken with respect to the "pivot" space $L^2(G)$ by continuity. Applying integration by parts, the mixed initial–boundary value problem for the scalar Wave equation with homogeneous Dirichlet or Neumann conditions can be written in the following variational form.

$$\begin{aligned} &\text{Find } u \in H^1(0,T_{\max};V), \text{ such that } \forall t \in (0,T_{\max}) \text{ and } \forall v \in V: \\ &\quad \partial_t^2(u(\cdot,t),v) + (\nabla u(\cdot,t),\nabla v) = (f(\cdot,t),v), \\ &\quad\quad (u(\cdot,0),v) = (u_0,v), \\ &\quad\quad \partial_t\,(u(\cdot,0),v) = (v_0,v), \end{aligned} \tag{2}$$

where $u_0 \in V$, $v_0 \in L^2(G)$ and where $f \in L^2(0,T_{\max};L^2(G))$ are given.

We discretize (2) by the method of lines, using continuous Lagrangian FEM of uniform polynomial degree $p \geq 1$ in the spatial domain G on a family of regular, simplicial triangulations of the domain G, followed by a non-specified discretization method in time. This is well-known to yield optimal convergence rates w.r.t. to the mesh size for the semi-discrete formulation, if $u \in C^2([0,T_{\max}];V^*)$. In convey domains, sufficient conditions for this to be satisfied are $f \in H^1(0,T_{\max};L^2(G)),\ u_0, v_0 \in V$, and the following compatibility conditions:

$$\frac{\partial^j}{\partial t^j}u(x,0) \in V, \quad j = 0,1,2,3, \quad \text{and} \quad \frac{\partial^4}{\partial t^4}u(x,0) \in L^2(G).$$

See, e.g., [14] for a detailed discussion of these compatibility conditions, where also necessary conditions for the regularity $u(\cdot,t) \in H^{p+1}(G)$ for domains G with smooth boundary are derived.

In the case the domain G is a generic bounded polygon in $\mathbb{R}^2$, higher regularity of $u(x,t)$ is only given in suitable scales of weighted Sobolev spaces, [9–11]. Therefore, further conditions on the mesh refinement need to be imposed. In Section 4, we will present two types of graded mesh refinements that approximate singular solutions with optimal convergence rates. Our main result will be given in Theorem 5.5 and states that for the space semi-discrete Finite Element approximation of the initial–boundary value problem of the scalar, second order Wave equation, the mesh families used in Section 4 yield optimal convergence rates. Hence, we consider the space semi-discrete case, and therefore our results are not restricted to specific time-stepping schemes. Numerical experiments which indicate that the theoretical estimates are sharp are presented in the last section. Throughout this paper, we use standard notation: the operators ∇ and Δ will be understood to only operate