Journal of Computational and Applied Mathematics 283 (2015) 182–200

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

# Revisit of a degenerate scale: A semi-circular disc

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ARTICLE INFO

*Article history:*
Received 25 June 2014
Received in revised form 9 January 2015

*Keywords:*
BEM
Degenerate scale
Logarithmic capacity
Bordered matrix
Singular value decomposition

ABSTRACT

Boundary element method (BEM) has been employed in engineering analysis since 1956, it has been widely applied in the engineering. However, the BEM/BIEM may result in an ill-conditioned system in some special situations, such as the degenerate scale. The degenerate scale also relates to the logarithmic capacity in the modern potential theory. In this paper, three indexes to detect the degenerate scale and five regularization techniques to circumvent the degenerate scale are reviewed and a new self-regularization technique by using the bordered matrix is proposed. Both the analytical study and the BEM implementation are addressed. For the analytical study, we employ the Riemann conformal mapping of complex variables to derive the unit logarithmic capacity. The degenerate scale can be analytically derived by using the conformal mapping as well as numerical detection by using the BEM. In the theoretical aspect, we prove that unit logarithmic capacity in the Riemann conformal mapping results in a degenerate scale. We revisit the Fredholm alternative theorem by using the singular value decomposition (SVD, the discrete system) and explain why the direct BEM and the indirect BEM are not indeed equivalent in the solution space. Besides, a zero index by using the free constant in Fichera's approach is also proposed to examine the degenerate scale. According to the relation between the SVD structure and Fichera's technique, we numerically provide a new self-regularization method in the matrix level. Finally, a semi-circular case and a special-shape case are designed to demonstrate the validity of six regularization techniques.



## 1. Introduction

Engineers employ the finite element method (FEM) and the finite difference method (FDM) to solve engineering problems by way of the model of partial differential equation (PDE), the unique solution can be numerically obtained for the Dirichlet problem or constrained structure. In 1956, Kinoshita and Mura [1] derived the singular boundary integral equation for the elasticity. The boundary element method (BEM) is easier than the FEM on the viewpoint of the discretization of one dimension reduction rather than the domain discretization of the FEM. Mathematical study of the boundary integral equation method (BIEM) and engineering applications of the BEM has been developed more than 50 years. However, the non-equivalence of the solution space in the BIE and the PDE was not noticed. Boundary value problems (BVP) can be solved by using different methods as shown in Fig. 1. Using the BEM to solve the Helmholtz equation or the Laplace equation, there are some rank-deficiency problems caused by fictitious frequency [2,3], spurious eigenvalue [4], degenerate boundary or

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http://dx.doi.org/10.1016/j.cam.2015.01.011
0377-0427/

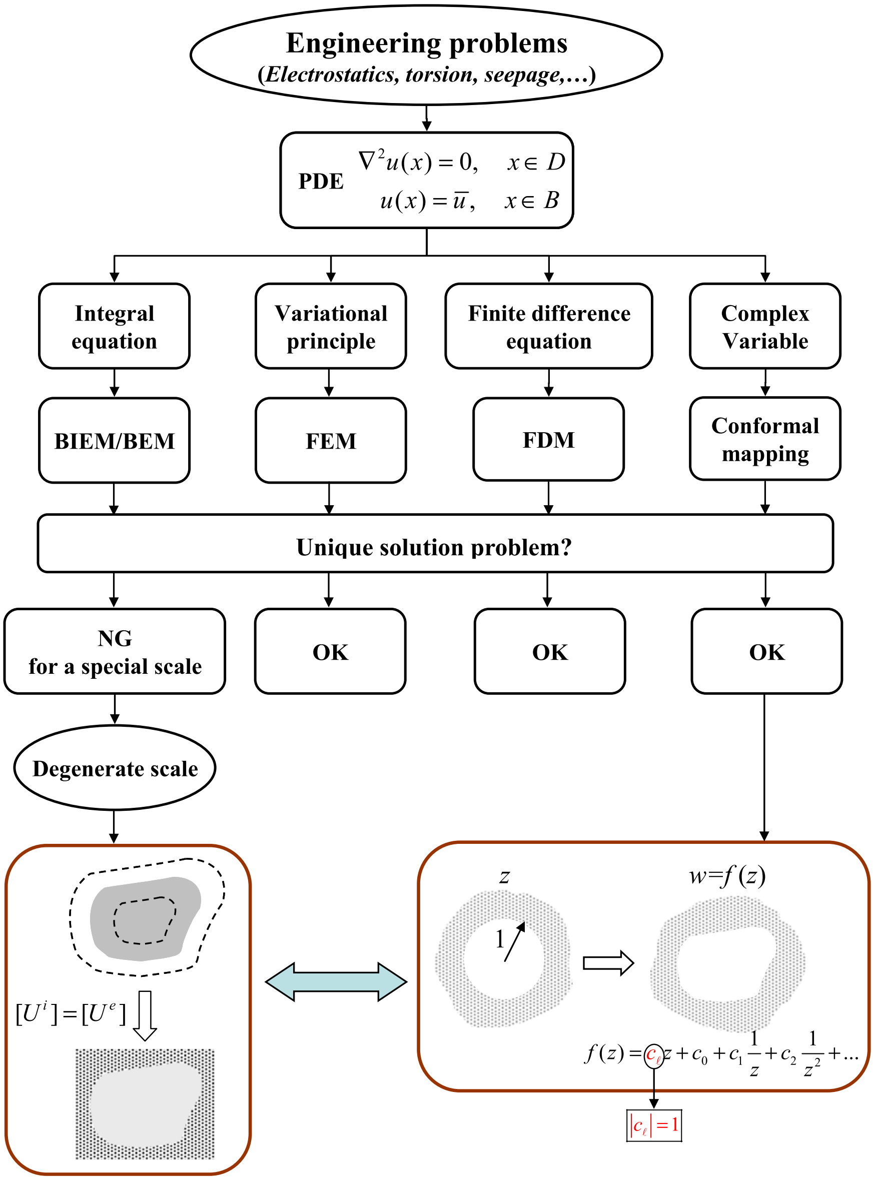


**Fig. 1.** Nonunique solution of numerical methods for 2-D Laplace problem.

degenerate scale [5–8]. Due to these pitfalls, the user of commercial codes of the BEM must be aware that numerical results may not be correct when the BEM is used. In this paper, we focus on the degenerate scale of 2D Laplace problems subject to the Dirichlet boundary condition. While using the BIEM or BEM to solve the 2D Dirichlet problem for a certain size, the non-uniqueness solution may appear. This critical size of the boundary is the degenerate scale which is also called the Gamma contour [9]. From the view point of mathematics, physics and numerics, the non-uniqueness solutions of 2D interior and exterior Laplace problems subject to various boundary conditions are listed in Table 1.

In structural dynamics, the concept of modal participation factor [10] is well known for structural engineers. It indicates the weighting how the corresponding mode contributes to the total response. This concept can be applied to the excitations of body force, boundary force and boundary support motion. A modal participation factor for the corresponding modes can be determined to predict the contribution of the numerical instability for the fictitious frequency in the BEM [11]. Based on this concept, it is easily understood that the concentrated symmetric load in Fig. 2(a) cannot excite the anti-symmetric mode in Fig. 2(b). To understand the mathematical problem of the degenerate scale in a similar way, we can expand the fundamental solution into superposition of ring force as shown in Fig. 2(c) through the addition theorem (degenerate kernel). A point source system of fundamental solution is an auxiliary system for Green's third identity. Another system is the considered Dirichlet problem as shown in Fig. 2(d). By considering the Green's third identity with respect to the two systems, we find that the unknown coefficient $p_0$ cannot be determined due to the zero work of boundary force of the considered system in