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Coincidence and fixed points for multi-valued mappings and its application to nonconvex integral inclusions

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ABSTRACT

In this paper, we consider some problems on coincidence point and fixed point theorems for multi-valued mappings. Applying the characterizations of $\mathcal{P}$-functions, we establish some new existence theorems for coincidence point and fixed point distinct from Nadler's fixed point theorem, Berinde–Berinde's fixed point theorem, Mizoguchi–Takahashi's fixed point theorem and Du's fixed point theorem for nonlinear multi-valued contractive mappings in complete metric spaces. Our results compliment and extend the main results given by some authors in the literature. In the sequel, we consider a nonconvex integral inclusion and prove the Filippov type existence theorem by using an appropriate norm on the space of selection of a multi-function and a multi-valued contraction for set-valued mappings.

1. Introduction

Let (X, d) be a metric space. The celebrated Banach contraction theorem (Banach [1], 1922) assures us of a unique fixed point if a mapping $T : X \to X$ is a contraction, i.e., if there exists a positive number $\alpha < 1$ such that

$$d(Tx, Ty) \le \alpha d(x, y)$$

for all $x, y \in X$. The Banach contraction theorem formulated for a complete metric space is one of the most simple and, at the same time, the most important method for the existence and uniqueness of solution of nonlinear problems arising in mathematics and its applications to engineering and natural sciences.

In 1972, Chaterjea [2] introduced a contraction type mapping $T : X \to X$ by means of the metric relation

$$d(Tx, Ty) \le \alpha(d(x, Ty) + d(y, Tx))$$

for all $x, y \in X$ and $0 < \alpha < \frac{1}{2}$.

In 1969, Nadler [3] proved a multi-valued extension of the Banach contraction theorem. Nadler's contraction principle has led to fixed point theory of multi-valued contraction in nonlinear analysis. Inspiring from the results of Nadler the fixed

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point theory of multi-valued contraction was further developed in different directions by many authors, in particular, by Reich [4], Berinde–Berinde [5], Mizoguchi and Takahashi [6], Du [7–10], Ding et al. [11], Long et al. [12] and many others.

In the present paper, using the concept of generalized contractions for multi-valued mappings in metric spaces, we show that the existence of fixed point for such a contraction is guaranteed under certain conditions. Our first results are essentially compliment and distinct from Nadler's, Berinde–Berinde's, Mizoguchi–Takahashi's and Du's theorems.

In a parallel development, the arena of applied mathematics witnessed sparkling changes that took place while handling some real world problems. In this context, it is interesting to note that dynamical systems described by differential equations with continuous right-hand sides were the areas of vigorous steady in the later half of the 20th century, in particular, in the study of viscous fluid motion in a porous medium, propagation of light in an optically non-homogeneous medium, determining the shape of a solid of revolution moving in a flow of gas with least resistance, etc. Euler's equation plays a key role in dealing with the existence of the solution of such problems. On the other hand, Filippov [13] has developed a solution concept for differential equations with a discontinuous right-hand side. In practice, such dynamical systems do arise and require analysis. Examples of such systems are mechanical systems with Coulomb friction modeled as a force proportional to the sign of a velocity, systems whose control laws have discontinuities.

In connection to fixed point theorems for multi-valued mappings, Petrusel has discussed several operational inclusions in [14]. We initiate our discussion by introducing some preliminaries and notations.

2. Preliminaries and definitions

Let (X, d) be a metric space. We denote by $\mathcal{N}(X)$ the class of all nonempty subsets of X, by $\mathcal{CB}(X)$ the class of all nonempty closed and bounded subsets of X and $\mathcal{K}(X)$ the class of all nonempty compact subsets of X.

For any $A, B \in \mathcal{CB}(X)$, let $\mathcal{H} : \mathcal{CB}(X) \times \mathcal{CB}(X) \to \mathbb{R}$ be a mapping defined by

$$\mathcal{H}(A, B) = \max\left\{\sup_{x\in B} d(x, A),\ \sup_{y\in A} d(y, B)\right\},$$

where $d(x, A) = \inf_{y\in A} d(x, y)$. A mapping $\mathcal{H}$ is called *Hausdorff metric* induced by d.

Let $f, g : X \to X$ be self-mappings and $T : X \to \mathcal{N}(X)$ be a multi-valued map. A point x in X is a coincidence point of f, g and T if $fx = gx \in Tx$. If $f = g = id$ is the identity mapping, then $x = fx = gx \in Tx$ and call x a fixed point of T. The set of fixed points of T and the set of coincidence point of f, g and T are denoted by $\mathcal{F}(T)$ and $\mathcal{COP}(f, g, T)$, respectively.

Definition 2.1 (*[7–10]*)**.** A function $\varphi_{\mathcal{MT}} : (0, \infty) \to [0, 1)$ is called an $\mathcal{MT}$*-function* if it satisfies Mizoguchi–Takahashi's condition, that is,

$$\limsup_{r\to t^+} \varphi_{\mathcal{MT}}(r) < 1$$

for each $t \in [0, \infty)$.

Definition 2.2. A function $\varphi : (0, \infty) \to [0, \frac{1}{2})$ is called a $\mathcal{P}$*-function* if it satisfies the following condition:

$$\limsup_{r\to t^+} \varphi(r) < \frac{1}{2}$$

for each $t \in [0, \infty)$.

Definition 2.3. $\varphi : (0, \infty) \to [0, \frac{1}{2})$ is called a *function of semi-contractive factor* if, for any sequence $\{x_n\}$ in $[0, \infty)$, from and after some fixed term, it is a strictly decreasing sequence and

$$0 \leq \sup_{n\in\mathbb{N}} \varphi(x_n) < \frac{1}{2}.$$

The following example realizes the existence of $\mathcal{P}$-function and function of semi-contractive factor.

Example 2.4. Consider a sequence $\{x_n\}$ in $[0, \infty)$ given by

$$x_n = \begin{cases} 2^n - 1, & \text{if } n \leq 5, \\ 1 - 2^{-n}, & \text{if } n > 5, \end{cases}$$

and define a function $\varphi : (0, \infty) \to [0, \frac{1}{2})$ by

$$\varphi(t) = \begin{cases} \dfrac{1}{3+t}, & \text{if } 0 < t < 1, \\ \dfrac{1}{2} - \dfrac{t}{2^6}, & \text{if } 1 \leq t < 32, \\ 0, & \text{if } t \geq 32. \end{cases}$$