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# Extending fundamental formulas from classical B-splines to quantum B-splines

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ABSTRACT

We derive a collection of fundamental formulas for quantum B-splines analogous to known fundamental formulas for classical B-splines. Starting from known recursive formulas for evaluation and quantum differentiation along with quantum analogues of the Marsden identity, we derive quantum analogues of the de Boor–Fix formula for the dual functionals, explicit formulas for the quantum B-splines in terms of divided differences of truncated power functions, formulas for computing divided differences of arbitrary functions by quantum integrating certain quantum derivatives of these functions with respect to the quantum B-splines, closed formulas for the quantum integral of the quantum B-splines over their support, and finally a $1/q$-convolution formula for uniform $q$-B-splines.



## 1. Introduction

Quantum splines combine the quantum calculus with the theory of splines. The classical theory of splines treats piecewise polynomials whose classical derivatives up to some order match at the joins. The quantum calculus deals with discrete derivatives, that is, with divided differences. There are two common types of quantum derivatives – $h$-derivatives and $q$-derivatives – corresponding to $h$-divided differences and $q$-divided differences (see Section 2.2). Quantum splines are piecewise polynomials whose quantum derivatives up to some order agree at the joins. Thus quantum splines allow us to model tolerances, jumps, and even quantum leaps in the derivatives at the joins.

Quantum splines defined using the $h$-derivative were first introduced in [1,2] as solutions to certain minimization problems involving differences. Many properties of these $h$-splines are derived in [3]. The basic theory of $q$-splines was first presented in [4] based on $q$-blossoming. In their work several identities such as the $q$-Marsden Identity, as well as knot insertion algorithms and spline interpolation formulas are derived. Classical B-splines and quantum B-splines share many common properties. For example, each has compact support and each forms a partition of unity.

The goal of this paper is to extend several fundamental formulas for classical B-splines to quantum B-splines. We shall derive quantum analogues of the de Boor–Fix formula for the dual functionals, explicit formulas for the quantum B-splines in

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terms of divided differences of truncated power functions, formulas for computing divided differences of arbitrary functions by quantum integrating certain quantum derivatives of these functions with respect to the quantum B-splines, a closed formula for the quantum integral of the quantum B-splines over their support, and a $1/q$-convolution formula for uniform $q$-B-splines. For clarity and ease of comparison, in each case we shall state three formulas – the formula for classical B-splines, the corresponding formula for $h$-B-splines, and the corresponding formula for $q$-B-splines – one right after the other, but we shall prove only the formulas for $q$-B-splines. The proofs for the classical B-splines are already known, and the proofs for $h$-B-splines are typically quite similar to the proofs for the $q$-B-splines and therefore are left as simple exercises for the reader.

Since formulas for quantum B-splines employ quantum derivatives and quantum integrals, we begin in Section 2 with a brief review of the quantum calculus. To set the stage for the quantum calculus, we also provide a succinct review of divided differences, including only those properties of divided differences that we will employ later in this paper. In Section 3, we review three previously derived formulas and identities for classical and quantum B-splines: the two-term recurrences for the B-splines, the two-term formulas for derivatives of the B-splines in terms of lower order B-splines, and the Marsden Identities. These three formulas for the $h$-B-splines are derived in [3], using basic properties of divided differences; the corresponding three formulas for the $q$-B-splines are derived in [4], based mainly on properties of the $q$-blossom. These three formulas will each play a fundamental role in our derivations in Sections 4 and 5 of new formulas and identities for the quantum B-splines.

Sections 4 and 5 contain the main results of this paper. Here we extend five standard formulas from classical B-splines to quantum B-splines: the de Boor–Fix formula for the dual functionals, the explicit formula for the B-splines in terms of the divided differences of truncated powers, the formula for arbitrary divided differences as integrals of products of B-splines with high order derivatives, the simple closed formulas for the integrals of the B-splines over their support, and finally the convolution formula for the uniform B-splines. Our proofs employ only standard results about divided differences and simple results from the quantum calculus, along with the three previously derived formulas for the quantum B-splines. In each case we compare our new formulas for quantum B-splines to the corresponding standard formulas the classical B-splines.

## 2. Quantum calculus

The primary goal of this paper is to derive formulas and identities for quantum B-splines. Since these formula involve quantum derivatives and quantum integrals, we provide here a short review of the quantum calculus. To learn more about the quantum calculus, see [5].

### 2.1. Divided differences

We begin with divided differences, since both classical and quantum derivatives can be defined in terms of divided differences. For more details, properties and proofs, see [6].

**Definition 1.** Let $F$ be an arbitrary function. Then the divided difference of $F$ at the nodes $t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n$ is defined recursively by

$$\begin{aligned} F[t_0] &= F(t_0) && \\ F[t_0, t_1] &= \frac{F(t_1) - F(t_0)}{t_1 - t_0} && t_1 \neq t_0 \\ &= F'(t_0) && t_1 = t_0 \\ &\vdots && \vdots \\ F[t_0, \ldots, t_n] &= \frac{F[t_1, \ldots, t_n] - F[t_0, \ldots, t_{n-1}]}{t_n - t_0} && t_n \neq t_0 \\ &= \frac{F^{(n)}(t_0)}{n!} && t_n = t_0. \end{aligned}$$

Notice that the divided difference is a linear operator. That is,

$$\begin{aligned} &(F + G)[t_0, \ldots, t_n] = F[t_0, \ldots, t_n] + G[t_0, \ldots, t_n] \\ &(cF)[t_0, \ldots, t_n] = c(F[t_0, \ldots, t_n]). \end{aligned} \tag{2.1}$$

There are simple closed formulas for the divided differences of low degree polynomials. We summarize these results in the following theorem.

**Theorem 1.** *Let $P(t)$ be a polynomial.*

(i) *If* $\deg(P(t)) \leqslant n - 1$, *then* $P[t_0, \ldots, t_n] = 0$.
(ii) *If* $\deg(P(t)) = n$, *then* $P[t_0, \ldots, t_n]$ *is the coefficient of* $t^n$ *in the monomial representation for* $P(t)$. *Thus in this case,* $P[t_0, \ldots, t_n]$ *is a constant independent of* $t_0, \ldots, t_n$.

We shall use the next theorem in our proof in Section 4.2 of the representation of the quantum B-splines in terms of divided differences of truncated power functions.