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Numerical integration of ordinary differential equations with rapidly oscillatory factors

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ABSTRACT

We present a methodology for numerically integrating ordinary differential equations containing rapidly oscillatory terms. This challenge is distinct from that for differential equations which have rapidly oscillatory solutions: here the differential equation itself has the oscillatory terms. Our method generalises Filon quadrature for integrals, and is analogous to integral techniques designed to solve stochastic differential equations and, as such, is applicable to a wide variety of ordinary differential equations with rapidly oscillating factors. The proposed method flexibly achieves varying levels of accuracy depending upon the truncation of the expansion of certain integrals. Users will choose the level of truncation to suit the parameter regime of interest in their numerical integration.

1. Introduction

Ordinary differential equations (ODEs) containing rapidly oscillatory terms are a challenge for numerical computation. A separate much researched challenge are ODEs where the equations are not themselves rapidly oscillatory, but do have rapidly oscillatory solutions. Here we focus on the case where the ODE contains both terms which rapidly oscillate on a microscale time and terms which vary smoothly over macroscale times of interest. The microscale oscillating terms combined with the slow macroscale terms in the ODE produce solutions with multiscale structure. Typically, solutions are smoothly varying over the macroscale, but with superimposed microscale detail (e.g., Fig. 1). Such microscale detail interacts via nonlinearity to modify the apparent macroscale behaviour.

We consider the class of ODEs for some function $u(t) \in \mathbb{R}^m$ of the form

$$\frac{du}{dt} = a(t, u) + b(t, u)v(t), \qquad u(t_n) = u_{t_n}, \tag{1}$$

for smoothly varying coefficient functions $a, b : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$, and where the 'vacillating' $v(t)$ is some given rapidly oscillating periodic scalar function of time t and constant oscillation frequency ϖ (that is, period $T = 2\pi\varpi^{-1}$). The rapidly oscillating $v(t)$ may be functions such as $\sin \varpi t$ or $e^{i\varpi t}$. Suppose we are interested in sampling the solution over a relatively long macroscale time, say over time steps of size h. We consider just one of these steps, from initial time t_n to final time $t_{n+1} = t_n + h$. We assume the microscale oscillation is rapid with respect to the macroscale time scale h so that $\varpi^{-1} \ll h$. For definiteness, we also assume time t and unknown u have been scaled so that the coefficient functions a and b vary on a scale of one in both t and u. Fig. 1 plots solutions from the example ODE (39), discussed in Section 3, that are in the class (1) of the ODEs considered here. In these examples the multiscale structure of the solution is clearly visible. Over the macroscale time interval [0, 1] the general trend of the solution is revealed, but over microscale time intervals of the order $\varpi^{-1} = 0.01$ the solution is highly oscillatory.

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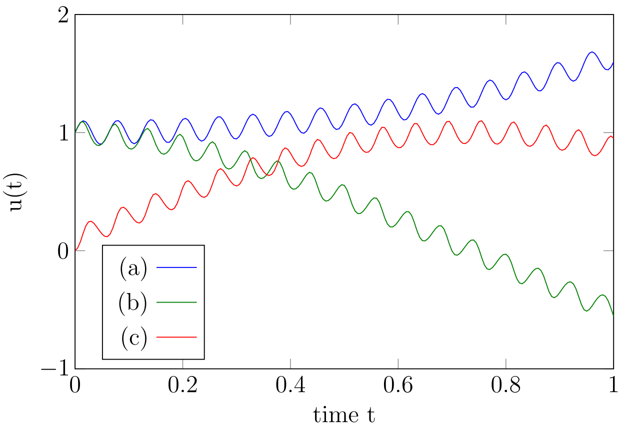

Fig. 1. Example solutions, using microscale time steps, of the ODE (39) for oscillations of strength $\mu = 10$ and frequency $\varpi = 100$, and initial condition $u(0) = 1$: (a) $\gamma = 0, \alpha(t) = t, v(t) = \cos \varpi t$; (b) real part for $\gamma = 2, \alpha(t) = 2i, v(t) = e^{i\varpi t}$; and (c) imaginary part for $\gamma = 2, \alpha(t) = 2i, v(t) = e^{i\varpi t}$.

The ODE (1) arises in a wide variety of systems with diverse applications. For example, rapidly oscillating external electric and magnetic fields [1] or external optical fields (produced by lasers) [2] create particle traps in which the state of the trapped particles (e.g., ions, atoms) is controllable. In communication systems, a slowly varying signal containing information is applied to a rapidly oscillating carrier signal [3]. In the a.c. Josephson effect, an applied constant voltage across a Josephson junction (of either superconductors or Bose–Einstein condensates) causes an alternating current $\dot{\eta}$ to flow across the junction with frequency dependent on the chemical potential difference between the materials on either side of the junction and η the relative atomic density across the junction [4]. Second order ODE with rapidly oscillating terms include the seemingly simple model of a rigid pendulum with an oscillating suspension point [5,6], and the more general forced van der Pol–Mathieu–Duffing oscillator [7,8], which provide insights into multiple modes and chaotic behaviour.

Established numerical techniques, such as Runge–Kutta or Gear's method, work well for ODEs without rapidly oscillating terms. However, these techniques become computationally expensive when oscillations with microscale periods of order ϖ^{-1} are present, particularly when there is a significant difference between the two relevant time scales, $\varpi^{-1} \ll h$. For example, Matlab's stiff ODE solver `ode15s` takes 15 time steps per microscale oscillation to reproduce the ODE solutions shown in Fig. 1. For this reason, specialised numerical methods are required to accurately and efficiently evaluate ODEs containing rapidly oscillating terms [9–12]. A closely related problem is the numerical evaluation of integrals with rapidly oscillating integrands relative to the range of integration. Such integrals are computationally expensive to solve using standard numerical integration methods, but are efficiently solved using specialised methods [9,13–15]. The aim herein is to develop an efficient and flexible computational scheme for the evaluation of ODE where each time step spans many microscale oscillation periods.

Numerous multiscale modelling techniques seek slowly varying macroscale solutions of ODE with microscale detail [16–18, for reviews]. The heterogeneous multiscale method (HMM) [19,20], for example, decomposes the original ODE into two ODEs, one which contains all information concerning fast dynamics and one which describes only slow dynamics. The slow ODE is solved to obtain a macroscale solution, but with reference to the fast ODE in regions where the slow ODE solution is incomplete or invalid. The HMM is effective in solving ODE of the form $du/dt = f_0(t, u) + \epsilon^{-1} f_1(t, u)$ with small time scale parameter ϵ and functions f_0 and f_1 which contain oscillating components [21,22]. This ODE which is amenable to the HMM produces a *rapidly oscillating solution* u, since du/dt has a large range when ϵ is small, but the ODE itself contains *no rapidly oscillating terms* since $f_{0,1}$ are not explicitly dependent on the small scale ϵ. In contrast, we are interested in ODE such as (1) which contains oscillating functions with small periods which are explicitly of the order of the small time scale ϖ^{-1}. Many multiscale methods have, like HMM, been applied to ODE with rapidly oscillating solutions, but few address ODE (1) with rapidly oscillating terms. One exception is the method by Condon, Deaño and Iserles [10–12] discussed in Section 3.2 as a comparison to our proposed method.

Our novel method for numerically solving ODEs with rapidly oscillating terms is based upon iterating integrals. Section 2 derives a multivariable Taylor series expansion for the solution at time t_{n+1}, based about the solution at time t_n in powers of the typical microscale period of oscillation ϖ^{-1} and the macroscale time step $h = (t_{n+1} - t_n)$ [23]. The general form of the expansion is given in Proposition 5. Validity of the Taylor expansion requires small ϖ^{-1} and h, typically $\varpi^{-1}, h < 1$. The integral approach empowers us to quantify the remainder term in the Taylor expansion, and hence empowers users to potentially bound the errors in any application of the approximation scheme. In these systems, the macroscale time step h is much longer than a typical microscale period of oscillation ϖ^{-1}, so $h\varpi \gg 1$. The method is analogous to a Taylor expansion scheme, originally developed for Ito stochastic differential equations (SDEs) governed by a Wiener process, which is achieved by an iterative application of the Ito formula [23,24].

A typical Ito SDE,

$$du = a(t, u)dt + b(t, u)dW_t, \tag{2}$$