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Letter to the editor

Precise and fast computation of complete elliptic integrals by piecewise minimax rational function approximation

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ABSTRACT

Piecewise minimax rational function approximations with the single and double precision accuracies are developed for (i) $K(m)$ and $E(m)$, the complete elliptic integral of the first and second kind, respectively, and (ii) $B(m) \equiv (E(m) - (1-m)K(m))/m$ and $D(m) \equiv (K(m) - E(m))/m$, two associate complete elliptic integrals of the second kind. The maximum relative error is one and 5 machine epsilons in the single and double precision computations, respectively. The new approximations run faster than the exponential function. When compared with the previous methods (Fukushima, 2009; Fukushima, 2011), which have been the fastest among the existing double precision procedures, the new method requires around a half of the memory and runs 1.7–2.2 times faster.

1. Introduction

The Legendre normal form complete elliptic integrals of the first and second kind, $K(m)$ and $E(m)$, and their linear combinations, $B(m)$ and $D(m)$, appear in various scenes of science and technology [1, Introduction]. They are defined [2, Section 4.4] as

$$K(m) \equiv \int_0^{\pi/2} \frac{d\theta}{\Delta(\theta|m)}, \qquad E(m) \equiv \int_0^{\pi/2} \Delta(\theta|m) d\theta, \tag{1}$$

$$B(m) \equiv \int_0^{\pi/2} \frac{\cos^2\theta d\theta}{\Delta(\theta|m)}, \qquad D(m) \equiv \int_0^{\pi/2} \frac{\sin^2\theta d\theta}{\Delta(\theta|m)}, \tag{2}$$

where m is the parameter [3, Section 3.2.11] and $\Delta(\theta|m)$ is Jacobi's Delta function defined as

$$\Delta(\theta|m) \equiv \sqrt{1 - m\sin^2\theta}. \tag{3}$$

Refer to Fig. 1 for the behavior of these integrals. The standard reference of elliptic integrals is [4, Chapter 19], which is freely accessible at http://dlmf.nist.gov/.

Among the existing methods to compute these integrals, $K(m)$, $E(m)$, $B(m)$, and $D(m)$, the fastest procedures are those based on the piecewise truncated Taylor series expansions [5,6] as shown in [2, Table 1]. Nevertheless, their relative error curves are far from being minimax because the adopted approximations are obtained by truncating the Taylor series expanded around the mid point of each sub interval. Also, the sub intervals are not optimally chosen. In fact, their separation points are naively set as 0.1, 0.2, . . . , 0.8, 0.85, and 0.9 [5,6]. This was just for simplicity.

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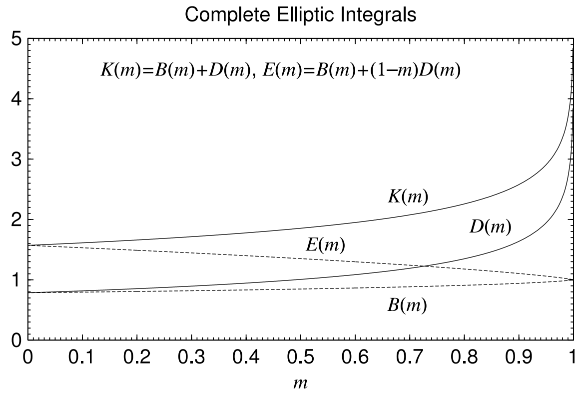

Fig. 1. Sketch of four complete elliptic integrals.

Table 1
Separation points of piecewise minimax rational approximations. Listed are the separation points of the sub intervals of approximation, m_j, such that the jth sub interval becomes $[m_{j-1}, m_j)$ while $m_0 \equiv 0$.

Precision	j	$K(m)$	$E(m)$	$B(m)$	$D(m)$
Single	1	0.734599	0.791217	0.810113	0.721197
	2	0.931503	0.968607	0.974420	0.924036
	3	0.982854	0.997566	0.998303	0.979874
Double	1	0.407010	0.433362	0.444927	0.400091
	2	0.649244	0.684847	0.697633	0.640820
	3	0.793076	0.828645	0.838948	0.785426
	4	0.878266	0.909330	0.916478	0.872125
	5	0.928558	0.953547	0.958034	0.923993
	6	0.958230	0.977087	0.979687	0.954948
	7	0.975640	0.989191	0.990592	0.973374
	8	0.985835	0.995159		0.984311
	9	0.991787			0.990784

In general, rational function approximations are better than polynomial ones including the truncated Taylor series in the sense to have a higher computational cost performance [7, Chapter 5]. Also, the recent computer chips such as the Intel Core series are capable to execute at least two multiply-and-add operations in parallel [8]. This fact enhances the cost performance of the evaluation of rational functions, especially of the even type, since the numerator and denominator polynomials can be evaluated by Horner's method simultaneously.

This short report provides a set of new formulas to compute these complete integrals by using the minimax rational function approximation. Section 2 describes the process of their construction, the summary of the obtained results, and the computational cost and performance of the new formulas.

2. Results

Construct a piecewise minimax approximation of the integrals, $K(m)$, $E(m)$, $B(m)$, and $D(m)$, when $0 \le m < 1$. Following the previous methods [5,6], the standard interval, $0 \le m < 1$, is split into several regions as shown in Table 1. The number of the sub intervals and the separation points are experimentally determined in order to take a balance between the memory saving and the computational speed-up while letting the relative error curves globally minimax as will be shown later.

Except the last sub interval, the integrals are approximated by rational functions of an even type as

$$K(m) \approx K_j(m) \equiv \frac{\sum_{n=0}^{N} P_n t^n}{\sum_{n=0}^{N} Q_n t^n}, \quad \left(m_{j-1} \le m < m_j;\ j = 1, \ldots, J\right) \tag{4}$$

where N is set as 3 and 5 for the single and double precision environments, respectively, and

$$t \equiv \frac{(m_j - 1) + m_c}{m_j - m_{j-1}}, \tag{5}$$