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# A novel method for nonlinear boundary value problems

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ABSTRACT

In this paper, a novel numerical method is proposed for solving nonlinear boundary value problems. This method is based on a combination of the reproducing kernel and least squares methods. So the accuracy of this method is improved. We have proved the uniform convergence of the approximate solution $u_n$ by convergence of $u_n$ in $W_3[a, b]$. Numerical results obtained by using the scheme show that the numerical scheme is feasible and effective.



## 1. Introduction

In this paper, we consider the following nonlinear boundary value problem:

$$\begin{cases} a_2(x)u'' + a_1(x)u' + a_0(x)u + g(x, u) = f(x), & a < x < b \\ u(a) = \alpha_1, \qquad u(b) = \alpha_2, \end{cases} \tag{1.1}$$

where $a_i(x), f(x) \in C[a, b], \alpha_i \in \mathbb{R}, g(x, y) \in C([a, b] \times \mathbb{R})$.

Nonlinear boundary value problems arise in the mathematical modeling of viscoelastic and inelastic flows, deformation of beams and plate deflection theory [1–3]. The research on these models is much more active in recent years.

Nonlinear boundary value problems have been treated extensively by homotopy perturbation method, successive interpolations method, a semi-analytic technique for generating smooth nonuniform grids, variational approach, cubic spline, combined homotopy perturbation method and Green's function method in recent years [4–10].

In recent years, some researches on the multi-point boundary value problem and initial value problem algorithms have been proposed, such as [11–19]. These papers in the literature have provided a number of numerical simulation, and show people's increasing demand for the feasibility and convergence of the algorithm.

In this paper, a novel numerical method is proposed for solving nonlinear boundary value problems. This method is based on a combination of the reproducing kernel and least squares methods. We have proved the uniform convergence of the approximate solution $u_n$ by convergence of $u_n$ in $W_3[a, b]$. So that this method has got higher accuracy. Numerical results obtained by using the scheme show that the numerical scheme is feasible and effective. On the issue of (1.1), we propose this method. The method is also applied to solve other models, such as nonlinear models of higher order, complicated boundary value problems and so on.

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## 2. Preliminaries

Reproducing kernel space $W_3[a,b]$ is defined as $W_3[a,b] = \{u(x)|u''(x)$ is an absolutely continuous real value function, $u'''(x) \in L^2[a,b]\}$. The inner product in $W_3[a,b]$ is given by

$$\langle u(x), v(x)\rangle = u(a)v(a) + u'(a)v'(a) + u''(a)v''(a) + \int_a^b u'''v'''\mathrm{dy}.$$

Its reproducing kernel is $R_x(y)$ [20].

Reproducing kernel space $W_1[a,b]$ is defined by $W_1[a,b] = \{u(x)|u$ is an absolutely continuous real valued function, $u' \in L^2[a,b]\}$. The inner product and norm in $W_1[a,b]$ are given respectively by

$$\langle u(x), v(x)\rangle = u(a)v(a) + \int_a^b u'(y)v'(y)\mathrm{dy}, \quad \|u\| = \sqrt{\langle u, u\rangle}.$$

Its reproducing kernel is denoted by

$$r_x(y) = \begin{cases} 1 + y - a, & x \geq y, \\ 1 + x - a, & x < y. \end{cases}$$

In order to solve Eq. (1.1), we introduce a linear operator $L : W_3 \to W_1$

$$Lu(x) = a_2(x)u'' + a_1(x)u' + a_0(x)u.$$

It is easy to prove that $L : W_3[a,b] \to W_1[a,b]$ is a bounded linear operator.

Then Eq. (1.1) can be transformed into the following form:

$$\begin{cases} Lu(x) = f(x) - g(x,u), & a < x < b, \\ u(a) = \alpha_1, \quad u(b) = \alpha_2. \end{cases} \tag{2.1}$$

Put

$$\varphi_1(x) = R_a(x), \qquad \varphi_2(x) = R_b(x) \in W_3, \tag{2.2}$$

and $\psi_i(x) = L^* r_{x_i}(x)$, where $L^*$ is the adjoint operator of $L$, $x_i \in [a,b]$.

**Lemma 2.1.**

$$\psi_i(x) = LR_x(x_i). \tag{2.3}$$

**Proof.** $\psi_i(x) = \langle L^* r_{x_i}(\cdot), R_x(\cdot)\rangle_{W_3} = \langle r_{x_i}(\cdot), LR_x(\cdot)\rangle_{W_1} = LR_x(x_i)$. □

**Theorem 2.2.** *If $\{x_i\}_{i=1}^{\infty}$ is dense on $[a,b]$, then for each fixed $n$, $\{\varphi_1(x), \varphi_2(x)\} \cup \{\psi_i(x)\}_{1\leq i\leq n}$ are linearly independent on $W_3[a,b]$.*

**Proof.** Let $0 = \sum_{i=1}^{n} \lambda_i\psi_i + k_1\varphi_1 + k_2\varphi_2$,

(1) Take $v_j \in W_3 (j = 1, \ldots, n)$, make

$$Lv_j = f_j(t) = \frac{\prod\limits_{\substack{i=0,1,\ldots,n \\ i\neq j}} (t - t_i)}{\prod\limits_{\substack{i=0,1,\ldots,n \\ i\neq j}} (t_j - t_i)}, \quad v_j(a) = 0, v_j(b) = 0,$$

then

$$\begin{aligned} 0 &= \left\langle v_j(t), \sum_{i=1}^{n} \lambda_i\psi_i + k_1\varphi_1 + k_2\varphi_2 \right\rangle \\ &= \sum_{i=1}^{n} \lambda_i \langle v_j(t), L^* r_{t_i}\rangle + k_1\langle v_j(t), R_a(t)\rangle + k_2\langle v_j(t), R_b(t)\rangle \\ &= \sum_{i=1}^{n} \lambda_i \langle Lv_j(t), r_{t_i}\rangle + k_1 v_j(a) + k_2 v_j(b) = \sum_{i=1}^{n} \lambda_i f_j(t_i) + 0 = \lambda_j. \end{aligned}$$