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Minimal asymptotic error for one-point approximation of SDEs with time-irregular coefficients ☆

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ABSTRACT

We consider strong one-point approximation of solutions of scalar stochastic differential equations (SDEs) with irregular coefficients. The drift coefficient $a : [0, T] \times \mathbb{R} \to \mathbb{R}$ is assumed to be Lipschitz continuous with respect to the space variable but only measurable with respect to the time variable. For the diffusion coefficient $b : [0, T] \to \mathbb{R}$ we assume that it is only piecewise Hölder continuous with Hölder exponent $\varrho \in (0, 1]$. We show that, roughly speaking, the error of any algorithm, which uses n values of the diffusion coefficient, cannot converge to zero faster than $n^{-\min\{\varrho,1/2\}}$ as $n \to +\infty$. This best speed of convergence is achieved by the randomized Euler scheme.

1. Introduction

In the paper we deal with one-point approximation of scalar stochastic differential equations of the form

$$\begin{cases} dX(t) = a(t, X(t))dt + b(t)dW(t), & t \in [0, T], \\ X(0) = \eta, \end{cases} \tag{1}$$

where $T > 0$, $\{W(t)\}_{t\in[0,T]}$ is a standard one-dimensional Brownian motion on some probability space $(\Omega, \Sigma, \mathbb{P})$ and the initial-value η is independent of $\{W(t)\}_{t\in[0,T]}$. In order to assure that (1) has a unique strong solution we assume that the drift coefficient $a : [0, T] \times \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous in $\mathbb{R}$ with respect to the space variable. However, we assume that the functions $a = a(t, y)$ and $b = b(t)$ suffer from lack of regularity with respect to the time variable t. This means that a is only *measurable* with respect to t, while b is only *piecewise Hölder continuous* in $[0, T]$ with Hölder exponent $\varrho \in (0, 1]$. Since analytic solutions of equations with such coefficients are known only in particular cases, the efficient approximation with the (asymptotic) error as small as possible is of interest.

The approximation of ordinary differential equations (ODEs i.e., problem (1) with $b \equiv 0$) with a non-smooth right-hand side function a has been investigated in the literature. For instance, ODEs with Carathéodory coefficients a were considered in [1–3]. In that papers the Monte Carlo methods were used to approximate the solutions. In [4] the authors investigated systems of initial-value problems with right-hand side functions a that have discontinuous partial derivatives. They proposed a version of Taylor algorithm which handles singularities of a and has the same order of convergence as classical methods in the regular case. Optimality of this algorithm has also been established. Moreover, in [5] a derivative-free

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scheme has been defined that approximates systems of singular initial-value problems and also preserves the optimal error known from the regular case.

The problem of approximating stochastic differential equations (SDEs) with regular coefficients a and b has been well investigated in the literature, see for example [6], which is the standard reference, and [7]. In contrast, much less is known about approximation of SDEs with discontinuous coefficients. In [8] weak convergence of the Euler scheme was investigated for SDEs with discontinuous a and b (in [8] the author also investigated the multiplicative noise case). However, the order of weak convergence has not been established. In [9] the authors investigated also the rate of weak convergence of the Euler scheme for irregular drift coefficient $a = a(t, y)$. Strong convergence of the Euler scheme applied to SDEs with discontinuous drift coefficients has been studied in [10], while strong convergence of the drift-implicit square-root Euler approximations has been established in [11]. Optimal rates of convergence of the Euler scheme for Eq. (1), with coefficients a and b having finite number of unknown discontinuities with respect to the time variable t, have been shown in [12]. In [13] the authors investigated the error of the randomized Euler scheme applied to (1) with non-smooth a and b.

In [13] the problem (1) was investigated in the *worst-case setting* with respect to a, b and η. The error of an algorithm was meant as the largest value of the averaged difference between the actual and computed solutions over a certain class of (a, b, η), and the rate of convergence of the error to zero was of interest. In this work we use a different approach. The error is now considered to be the averaged difference between the value of $X(T)$ and computed approximation for a fixed (arbitrary) (a, b, η) from a certain set of input data. The rate of convergence of the error to zero is then studied. We look for an algorithm from a certain class of algorithms whose speed of convergence is the best, for all (a, b, η). This is the so-called *asymptotic setting*. The main problem here is to establish lower bounds on the error of an arbitrary algorithm. We mean by that establishing the existence of a set of 'difficult' input data (a, b, η) for which the speed of convergence cannot be improved. We also want to know how 'large' a subset of such mappings (a, b, η) is. Results on lower bounds in the worst-case setting cannot be directly applied to obtain lower bounds in the asymptotic setting, see [14] and Chapter 10 in [15] for a further discussion. For a detailed description of our goal, see the next Section. An algorithm with the best convergence properties is sometimes referred to as the *minimal error algorithm*. For SDEs with smooth drift and diffusion coefficients the minimal asymptotic errors have been established in [7,16].

In this paper we investigate behavior of the asymptotic error for an arbitrary sequence of approximations $\hat{X}(T) = \{\hat{X}_n(T)\}_{n\in\mathbb{N}}$ of $X(T)$, where each $\hat{X}_n(T)$ uses n samples of the (piecewise smooth) diffusion coefficient b as $n \to +\infty$. We want to know the relation between the regularity of b and the rate of convergence of the algorithm.

Since we assume that the coefficients $a = a(t, y)$ and $b = b(t)$ do not have to be differentiable with respect to the time t and space y variables the lower bounds on the error developed in [7,16], for the one-point approximation of SDEs, cannot be directly applied to the class of (a, b, η) considered in this paper. Moreover, for upper bound, due to low regularity of a and b, we cannot use the algorithms developed there. In order to approximate the solution of (1) we use here the randomized Euler algorithm $\hat{X}^{RE}(T) = \{\hat{X}^{RE}_n(T)\}$, whose worst case error in the context of approximating SDEs was investigated in [13]. (See also [17,1–3] where the authors used suitable versions of $\hat{X}^{RE}(T)$ for approximation of solutions of ODEs.) In order to deliver corresponding asymptotic lower bounds under our assumptions we develop a suitable new technique. We use some general results developed in [18,14] and establish a new result concerning approximation of nonlinear operators defined on a subset of a space of bounded and measurable functions and with values in a normed space.

The main result of this paper states that the error of an arbitrary sequence of approximations $\{\hat{X}_n(T)\}_{n\in\mathbb{N}}$, where each $\hat{X}_n(T)$ uses n nonadaptive samples of a diffusion coefficient, cannot go to zero faster than $n^{-\min\{\varrho,1/2\}}$ as $n \to +\infty$ (Theorem 4.1). This holds except on the "small" subset of the set of b's under considerations. The "small" subset means here "of empty interior" (Theorem 3.2) or "of Lebesgue measure zero" (Theorem 3.4). Moreover, the randomized Euler scheme $\hat{X}^{RE}$ turns out to be optimal (Theorem 4.1).

The paper is organized as follows. Problem formulation and assumptions are given in Section 2. In Section 3 we establish lower bound of an arbitrary algorithm that uses nonadaptive or adaptive information about b. In Section 4 we recall the definition of the randomized Euler scheme $\hat{X}^{RE}(T)$ and discuss its error and optimality in the asymptotic setting. Finally, Appendix contains proofs of auxiliary lemmas.

2. Assumptions and basic definitions

Let $T > 0$ be a given real number. We denote $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $(\Omega, \Sigma, \mathbb{P})$ be a complete probability space and by $\mathcal{B}([0, T])$ we denote the σ-field of Borel subsets of $[0, T]$. By $\{\Sigma_t\}_{t\geq 0}$ we mean a filtration satisfying the usual conditions, such that the process $\{W(t)\}_{t\in[0,T]}$ is a Brownian motion on $(\Omega, \Sigma, \mathbb{P})$ with respect to $\{\Sigma_t\}_{t\in[0,T]}$, see [19]. We set $\Sigma_\infty = \sigma\left(\bigcup_{t\geq 0} \Sigma_t\right)$. For a random variable $X : \Omega \to \mathbb{R}$ we write $\|X\|_{L^q(\Omega)} = (\mathbb{E}|X|^q)^{1/q}$, $q \in [2, +\infty)$. We denote by $L^2([0, T])$ the set of all $\mathcal{B}([0, T])$-measurable functions $f : [0, T] \to \mathbb{R}$ for which $\|f\|_{L^2([0,T])} = \left(\int_0^T (f(t))^2 dt\right)^{1/2}$ is finite. By $\mathcal{M}_\infty([0, T])$ we denote a linear space of all $\mathcal{B}([0, T])$-measurable functions $f : [0, T] \to \mathbb{R}$ defined on $[0, T]$ such that $\|f\|_\infty = \sup_{t\in[0,T]} |f(t)| < +\infty$. For a given $\varrho \in (0, 1]$ by $\mathcal{C}^\varrho_0([0, T])$ we mean the Hölder space of functions in $\mathcal{M}_\infty([0, T])$, which is a Banach space under the Hölder norm $\|f\|_\varrho = \sup_{t\in[0,T]} |f(t)| + \sup_{t,s\in[0,T],s\neq t} \frac{|f(t)-f(s)|}{|t-s|^\varrho}$, see for example [20].