Journal of Computational and Applied Mathematics 282 (2015) 157–166

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

General constraint preconditioning iteration method for singular saddle-point problems

Ai-Li Yang *, Guo-Feng Zhang, Yu-Jiang Wu

School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, PR China

A R T I C L E   I N F O

Article history:
Received 7 September 2013
Received in revised form 10 June 2014

MSC:
65F08
65F10
65F20

Keywords:
Singular saddle-point problems
Moore–Penrose inverse
Constraint preconditioning
Convergence property
Iteration method

A B S T R A C T

For the singular saddle-point problems with nonsymmetric positive definite (1, 1) block, we present a general constraint preconditioning (GCP) iteration method based on a singular constraint preconditioner. Using the properties of the Moore–Penrose inverse, the convergence properties of the GCP iteration method are studied. In particular, for each of the two different choices of the (1, 1) block of the singular constraint preconditioner, a detailed convergence condition is derived by analyzing the spectrum of the iteration matrix. Numerical experiments are used to illustrate the theoretical results and examine the effectiveness of the GCP iteration method. Moreover, the preconditioning effects of the singular constraint preconditioner for restarted generalized minimum residual (GMRES) and quasi-minimal residual (QMR) methods are also tested.

1. Introduction

Consider the following large, sparse singular saddle-point problems

$$Ax := \begin{pmatrix} W & B^T \\ -B & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} = b, \tag{1.1}$$

where $W \in \mathbb{R}^{n\times n}$ is nonsymmetric positive definite and $B \in \mathbb{R}^{m\times n}$ is rank deficient, i.e., $\operatorname{rank}(B) < m \le n$, $b \in \mathbb{R}^{n+m}$ is a given vector in the range of saddle-point matrix $A \in \mathbb{R}^{(n+m)\times(n+m)}$. Such kind of linear systems arise in many application areas, such as computational fluid dynamics, computational genetics, mixed finite element approximation of elliptic partial differential equations, constrained optimization, optimal control, weighted least-squares problems, electronic networks, and computer graphics; see [1–4] and references therein.

When the saddle-point matrix A in (1.1) is nonsingular, which requires B being of full row rank, a number of iteration methods and preconditioning techniques, such as Uzawa-type methods [1,5,6], Krylov subspace methods [7,8], Hermitian and skew-Hermitian splitting (HSS) iteration methods [9–13], constraint preconditioners [14–16] and so on, have been proposed to approximate the unique solution of the nonsingular saddle-point problem (1.1). The comprehensive surveys can be found in Refs. [17,18]. Within these results, the constraint preconditioner of the form

$$M = \begin{pmatrix} P & B^T \\ -B & 0 \end{pmatrix}, \tag{1.2}$$

* Corresponding author. Tel.: +86 931 8912483; fax: +86 931 8912481.
E-mail addresses: yangaili@lzu.edu.cn, cmalyang@gmail.com (A.-L. Yang).

http://dx.doi.org/10.1016/j.cam.2014.12.042
0377-0427/

with P being positive definite was widely analyzed; see [15,19,20,14,16]. Based on this preconditioner M, Golub and Wathen [14] studied the following basic iteration scheme

$$x^{(k+1)} = x^{(k)} + M^{-1}(b - Ax^{(k)}). \tag{1.3}$$

We call this scheme constraint preconditioning iteration method if M is chosen to be the nonsingular constraint preconditioner (1.2). Let H and S be respectively the symmetric and the skew-symmetric parts of matrix W, i.e.,

$$W = H + S, \quad \text{with } H = \frac{1}{2}(W + W^T) \quad \text{and} \quad S = \frac{1}{2}(W - W^T).$$

The convergence properties of (1.3) were derived by Golub and Wathen [14] when matrix P in M is chosen to be a multiple of the symmetric part of W, i.e., $P = \omega H$ with $\omega > 0$. If W is not far from a symmetric matrix (i.e., $\|S\|/\|H\|$ is a small number), the preconditioner M with $P = \omega H$ is very efficient. However, as can be expected, performance of this preconditioner with symmetric P deteriorates when W is essentially nonsymmetric ($\|S\|/\|H\| \approx 1$ or larger). To overcome this deficiency, Botchev and Golub [19] proposed a novel constraint preconditioner M by choosing the $(1, 1)$ block of (1.2) as

$$P = \frac{1}{\omega}(I + \omega L_s)(I + \omega U_s), \tag{1.4}$$

where $\omega > 0$, L_s and U_s are, respectively, the lower and upper triangular parts of matrix S satisfying $L_s + U_s = S$ and $U_s = -L_s^T$. The preconditioner M with the new choice of P used for the iteration scheme (1.3) was proved to be efficient and robust for solving nonsingular saddle-point problems (1.1) with W being nonsymmetric. Moreover, as a preconditioner, it also can improve the convergence rate of GMRES method.

When B is rank deficient, saddle-point matrix A in (1.1) is singular. The linear systems (1.1) are called as singular saddle-point problems. Some iteration methods including Uzawa methods [4,21,22], Krylov subspace methods [3,23] and HSS iteration methods [24,25] have been used to solve this kind of singular problems. Since preconditioner M defined by (1.2) is also singular in this case, iteration scheme (1.3) cannot be used to solve singular saddle-point problems (1.1). In 2008, Cao [26] proposed an iteration scheme by replacing M^{-1} with $M^\dagger$ in (1.3) to solve general singular linear systems $Ax = b$, that is

$$x^{(k+1)} = x^{(k)} + M^\dagger(b - Ax^{(k)}), \tag{1.5}$$

where M is a singular matrix depending on the coefficient matrix A, $M^\dagger$ is the Moore–Penrose inverse of matrix M satisfying the following Moore–Penrose equations:

$$MM^\dagger M = M, \qquad (M^\dagger M)^* = M^\dagger M, \qquad (MM^\dagger)^* = MM^\dagger, \qquad M^\dagger MM^\dagger = M^\dagger. \tag{1.6}$$

Iteration scheme (1.5) was used later to solve the range-Hermitian singular linear systems by Zhang and Wei in [27], the numerical efficiencies of this method were also verified. We call iteration scheme (1.5) the general constraint preconditioning (GCP) iteration method if M is a singular constraint preconditioner of the form (1.2).

In this work, we are especially interested in the case that matrix B is rank deficient, which means the saddle-point matrix A in (1.1) and the constraint preconditioner M in (1.2) are both singular. We use GCP iteration method (1.5) to solve the singular saddle-point problems (1.1). The remainder part of this work is organized as follows. In Section 2, we give the convergence properties of GCP iteration method (1.5) with M being of the form (1.2) and P being any positive definite matrix. For each of the two different choices of the matrix P, i.e., $P = \omega H$ and $P = (1/\omega)(I + \omega L_s)(I + \omega U_s)$, a detailed condition that guarantees the convergence of the GCP iteration method is derived in Section 3. In Section 4, numerical results show that the GCP iteration method (1.5), no matter as a solver or as a preconditioner for GMRES(10) and QMR methods, is robust and efficient. Finally in Section 5, we end this work with a brief conclusion.

2. Convergence properties

In this section, we analyze the convergence properties of the GCP iteration method (1.5) with M being defined in (1.2) and P being positive definite (maybe not symmetric). First, we present the following convergence result of iteration scheme (1.5) with any singular matrix M:

Lemma 2.1 ([26,28]). *Iteration scheme (1.5) is convergent if and only if the following three conditions are fulfilled:*

1. *null* $(M^\dagger A) =$ *null* (A)*;*
2. *index* $(I - T) = 1$*, or equivalently, rank* $(I - T) =$ *rank* $((I - T)^2)$*, where* $T := I - M^\dagger A$ *is the iteration matrix of (1.5);*
3. $\gamma(T) = \max\{|\lambda| : \lambda \in \sigma(T) \setminus \{1\}\} < 1$*, where* $\sigma(T)$ *is the spectral set of matrix* T*.*

In the following subsections, we analyze the convergence properties of GCP iteration method (1.5), i.e., M is singular matrix of the form (1.2), according to the three conditions of Lemma 2.1.