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Local convergence for multi-point-parametric Chebyshev–Halley-type methods of high convergence order

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ABSTRACT

We present a local convergence analysis for general multi-point-Chebyshev–Halley-type methods (MMCHTM) of high convergence order in order to approximate a solution of an equation in a Banach space setting. MMCHTM includes earlier methods given by others as special cases. The convergence ball for a class of MMCHTM methods is obtained under weaker hypotheses than before. Numerical examples are also presented in this study.

1. Introduction

In this paper we are concerned with the problem of approximating a solution x^* of the equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y.

Many problems in Computational Sciences and other disciplines can be brought in a form like (1.1) using mathematical modeling [1–5]. The solutions of these equations can rarely be found in closed form. That is why most solution methods for these equations are iterative. The practice of Numerical Functional Analysis for finding such solutions is essentially connected to Newton-like methods [1–19]. Newton's method converges quadratically to x^* if the initial guess is close enough to the solution. Iterative methods of convergence order higher than two such as Chebyshev–Halley-type methods [2–5,11–16,18,19] require the evaluation of the second Fréchet-derivative, which is very expensive in general. However, there are integral equations where the second Fréchet-derivative is diagonal by blocks and inespensive [3,11–16] or for quadratic equations, the second Fréchet-derivative is constant. Moreover, in some applications involving stiff systems, high

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order methods are useful. That is why it is important to study the convergence of high-order methods. In particular, we introduce the multi-point, multi-parametric Chebyshev–Halley-type methods MMCHTM defined by

$$y_n = x_n - \left[\mu I + \frac{\alpha}{2} G(x_n) + G(x_n)^2 Q(G(x_n))\right] \Gamma_n F(x_n),$$
$$z_n = x_n - \frac{\beta}{2} \Gamma_n F(x_n), \tag{1.2}$$
$$x_{n+1} = y_n - \left[I + \gamma G(x_n) + \delta G(x_n)^2 + \frac{1}{2} \Gamma_n B(z_n) G(x_n) \Gamma_n F(y_n) + \lambda G(x_n)^3\right] \Gamma_n F(y_n), \quad \text{for each } n = 0, 1, 2, \ldots,$$

where x_0 is an initial point, I is the identity operator, $\Gamma_n = F'(x_n)^{-1}$, B is a bilinear operator, $G(x_n) = \Gamma_n B(z_n) \Gamma_n F(x_n)$, Q is a continuous operator and $\alpha, \beta, \gamma, \delta, \lambda, \mu$ are real parameters chosen to force convergence of the method. MMCHTM unifies many earlier methods:

If $\alpha = \beta = \gamma = \delta = \lambda = \mu = 0$, $B = 0$ and $Q = 0$, we obtain Newton's method [3–5,9,10] defined by

$$x_{n+1} = x_n - \Gamma_n F(x_n) \quad \text{for each } n = 0, 1, 2, \ldots. \tag{1.3}$$

If $\alpha = \beta = \gamma = \delta = \lambda = 0$, $B = 0$ and $Q = 0$, we obtain the cubically convergent two-step Newton method [2–5,9] defined by

$$y_n = x_n - \Gamma_n F(x_n)$$
$$x_{n+1} = y_n - \Gamma_n F(x_n) \quad \text{for each } n = 0, 1, 2, \ldots. \tag{1.4}$$

If $\alpha = \beta = \delta = \lambda = \mu = 0$, $\gamma = \frac{1}{2}$, $B(x) = F''(x)$ and $Q = 0$, we obtain the Chebyshev method [1,2,4,8,14–16,19] defined by

$$x_{n+1} = x_n - \left[I + \frac{1}{2} G(x_n)\right] \Gamma_n F(x_n) \quad \text{for each } n = 0, 1, 2, \ldots. \tag{1.5}$$

If $\alpha = \delta = \lambda = \mu = 1$, $\beta = \gamma = 0$, $Q = \frac{\xi}{2}$, for $\xi \in [-1, 1]$ and $B(x) = F''(x)$, we obtain the Chebyshev–Halley-type method [6,18] defined by

$$y_n = x_n - \left[I + \frac{1}{2} G(x_n) + \frac{\xi}{2} G(x_n)^2\right] \Gamma_n F(x_n),$$
$$x_{n+1} = y_n - \left[I + G(x_n) + G(x)^2 + \frac{1}{2} \Gamma_n F''(x_n) G(x_n) \Gamma_n F(x_n)\right] \Gamma_n F(x_n) \quad \text{for each } n = 0, 1, 2, \ldots. \tag{1.6}$$

If $\alpha = \beta = \delta = \mu = \gamma = 1$, $\lambda \in [0, 1]$ and $B(x) = F''(x)$, we obtain the multi-point Chebyshev–Halley-type method (MCHTM) [19] defined by

$$y_n = x_n - [I + G(x_n) + G(x_n)^2 Q(G(x_n))] \Gamma_n F(x_n),$$
$$z_n = x_n - \frac{1}{2} \Gamma_n F(x_n)$$
$$x_{n+1} = y_n - \left[I + G(x_n) + G(x)^2 + \frac{1}{2} \Gamma_n F''(z_n) G(x_n) \Gamma_n F(x_n) \lambda G(x_n)^3\right] \Gamma_n F(y_n) \quad \text{for each } n = 0, 1, 2, \ldots. \tag{1.7}$$

The study about convergence of iterative procedures is normally centered on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give criteria ensuring the convergence of the iterative procedures. While the local analysis is based on the information around a solution, to find estimates of the radii of convergence balls. There exist many studies which deal with the local and the semilocal convergence analysis of Newton-like methods such as [1–19]. In particular, the semilocal convergence of MMCHTM was given in [19] under $(\mathcal{C})$ conditions:

($\mathcal{C}_1$) There exist $x_0 \in D$ and a constant $\xi_0 > 0$ such that $\Gamma_0 = F'(x_0)^{-1} \in L(Y, X)$ and $\|\Gamma_0\| \le \xi_0$;
There exist constants $\eta, \xi_1, \xi_2 > 0$ such that

($\mathcal{C}_2$)

$$\|\Gamma_0 F'(x_0)\| \le \eta;$$

($\mathcal{C}_3$)

$$\|F''(x)\| \le \xi_1 \quad \text{for each } x \in D;$$