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Analysis of moving least squares approximation revisited

Davoud Mirzaei

Department of Mathematics, University of Isfahan, 81745-163, Isfahan, Iran

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ABSTRACT

In this article the error estimation of the moving least squares approximation is provided for functions in fractional order Sobolev spaces. The analysis presented in this paper extends the previous estimations and explains some unnoticed mathematical details. An application to Galerkin method for partial differential equations is also supplied.

1. Introduction

The *Moving Least Squares (MLS)* approximation was introduced in an early paper by Lancaster and Salkauskas [1] in 1981 with special cases going back to McLain [2,3] in 1974 and 1976 and to Shepard [4] in 1968. For other early studies we can mention the work of Farwig [5–7]. Since, in MLS one writes the value of the unknown function in terms of *scattered* data, it can be used as an approximation to span the trial space in meshless (or meshfree) methods. This approximation has found many applications in curve fitting and numerical solutions of partial differential equations since early nineties [8–11].

The error analysis of MLS approximation was provided by some authors, beginning with the work of Farwig [7] which is limited to a univariate case. The connection to Backus–Gilbert optimality was studied by Levin [12] in 1998, and later it was used by Wendland [13–15] in a more elaborated setting. In [16] the analysis is presented for smooth functions in $C^{m+1}(\Omega)\cap H^{m+1}(\Omega)$. Armentano and Durán [17] proved error estimates in L^∞ for the function and its first derivatives in one dimensional case. Afterward Armentano [18] generalized this to multi-dimensional cases but it is still restricted to "convex" domains and Sobolev spaces of order one. One can also find an estimation in [19] for reproducing kernel particle methods (which is related to the MLS approximation) for integer order Sobolev spaces. They assumed a constant bound for the norm of the inverse matrix (matrix A in text) and considered it for special cases in one dimension and first order approximations. Note that the role of this matrix is very crucial in analysis. The paper of Zuppa [20] is also limited to some specific situations. In [13,15] the analysis presented only for the function in classical function spaces. We can also mention the work of Melenk [21] where the theoretical and computational aspects of some meshless approximation methods, including MLS, are considered.

The collocation method based on the MLS approximation is called *finite point method*. An analysis for this method has been presented in [22]. Besides, an interpolating MLS is developed recently. For error analysis and applications to element-free Galerkin method see [23,24].

The present work is based on the theory of Wendland and extends all the above results to a general case. All mathematical details are provided, special care is taken near the boundary, and lower bound for the minimum eigenvalue of the MLS local matrix is derived in general case, independent of the mesh-size. Besides, the analysis is presented for functions in fractional order Sobolev spaces. Finally an application to Galerkin methods for elliptic PDEs is investigated.

E-mail address: d.mirzaei@sci.ui.ac.ir.

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2. MLS approximation

Let $\Omega \subset \mathbb{R}^d$, for positive integer d, be a nonempty and bounded set. In the next section, more conditions on Ω will be imposed. Assume,

$$X = \{x_1, x_2, \dots, x_N\} \subset \Omega,$$

is a set containing N scattered points, called *centers* or *data site*. Distribution of points should be well enough to pave the way for analysis.

Henceforth, we use $\mathbb{P}_m^d$, for $m \in \mathbb{N}_0 = \{n \in \mathbb{Z}, n \geqslant 0\}$, as the space of d-variable polynomials of degree at most m of dimension $Q = \binom{m+d}{d}$. A basis for this space is denoted by $\{p_1, \dots, p_Q\}$ or $\{p_\alpha\}_{0 \leqslant |\alpha| \leqslant m}$. As usual, $B(x, r)$ stands for the ball of radius r centered at x.

The MLS, as a meshless approximation method, provides an approximation $s_{u,X}$ of u in terms of values $u(x_j)$ at centers x_j by

$$u(x) \approx s_{u,X}(x) = \sum_{j=1}^{N} a_j(x) u(x_j), \quad x \in \Omega, \tag{2.1}$$

where a_j are *MLS shape functions* given by

$$a_j(x) = w(x, x_j) \sum_{k=1}^{Q} \lambda_k(x) p_k(x_j), \tag{2.2}$$

where the influence of the centers is governed by weight function $w_j(x) = w(x, x_j)$, which vanishes for arguments $x, x_j \in \Omega$ with $\|x - x_j\|_2$ greater than a certain threshold, say δ. Thus we can define $w_j(x) = \Phi((x - x_j)/\delta)$ where $\Phi : \mathbb{R}^d \to \mathbb{R}$ is a nonnegative function with support in the unit ball $B(0, 1)$. Coefficients $\lambda_k(x)$ are the unique solution of

$$\sum_{k=1}^{Q} \lambda_k(x) \sum_{j \in J(x)} w_j(x) p_k(x_j) p_\ell(x_j) = p_\ell(x), \quad 0 \leqslant \ell \leqslant Q, \tag{2.3}$$

where $J(x) = \{j : \|x - x_j\|_2 \leqslant \delta\}$ is the family of indices of points in the support of w. In vector form

$$\boldsymbol{a}(x) = W(x) P^T (P W(x) P^T)^{-1} \boldsymbol{p}(x),$$

where $W(x)$ is the diagonal matrix carrying the weights $w_j(x)$ on its diagonal, P is a $Q \times \#J(x)$ matrix of values $p_k(x_j), j \in J(x), 1 \leqslant k \leqslant Q$, and $\boldsymbol{p} = (p_1, \dots, p_Q)^T$. In MLS one finds the best approximation to u at point x, out of $\mathbb{P}_m^d$ with respect to a discrete ℓ^2 norm induced by a *moving* inner product, where the corresponding weight function depends not only on points x_j but also on the evaluation point x in question. Note that $A(x) = PW(x)P^T$ is a symmetric positive definite matrix for all $x \in \Omega$. More details can be found in Chapter 4 of [15].

In what follows we will assume that Φ is nonnegative and continuous on $\mathbb{R}^d$ and positive on the ball $B(0, 1/2)$. In many application we can assume that

$$\Phi(x) = \phi(\|x\|_2), \quad x \in \mathbb{R}^d,$$

meaning that Φ is a radial function. Here $\phi : [0, \infty) \to \mathbb{R}$ is positive on $[0, 1/2]$, supported in $[0, 1]$ and its even extension is nonnegative and continuous on $\mathbb{R}$.

If, further, ϕ is sufficiently smooth, derivatives of u are usually approximated by derivatives of $s_{u,X}$,

$$D^\alpha u \approx D^\alpha s_{u,X}(x) = \sum_{j=1}^{N} D^\alpha a_j(x) u(x_j), \quad x \in \Omega, \tag{2.4}$$

and they are called *standard derivatives*. They are different from *GMLS* or *diffuse derivatives* [25] which are not the aim of this paper.

3. Error estimation

Since error estimates will be established using a variety of Sobolev spaces, we introduce them now. Let $\Omega \subset \mathbb{R}^d$ be a domain. For $k \in \mathbb{N}_0$, and $p \in [1, \infty)$, we define the Sobolev space $W_p^k(\Omega)$ to consist of all u with distributional derivatives $D^\alpha u \in L^p(\Omega)$, $|\alpha| \leqslant k$. The (semi-)norms associated with these spaces are defined as

$$|u|_{W_p^k(\Omega)} := \left(\sum_{|\alpha|=k} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p}, \qquad \|u\|_{W_p^k(\Omega)} := \left(\sum_{|\alpha|\leqslant k} \|D^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p}.$$