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Rounding errors of partial derivatives of simple eigenvalues of the quadratic eigenvalue problem

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ABSTRACT

In this paper, we derive rounding errors of partial derivatives of a simple eigenvalue of the quadratic eigenvalue problem dependent on several parameters. We prove a series of lemmas and finally get theorems of rounding errors of both nonsymmetric and symmetric QEPs. Examples are given to show the validity of our theorems, and numerical results show that our rounding error is a very good upper bound estimation of the relative error of the eigenvalue.

1. Introduction

Consider the following *Quadratic Eigenvalue Problem* (QEP) dependent on several parameters

$$Q(p,\lambda)u = 0, \qquad v^\top Q(p,\lambda) = 0, \tag{1.1}$$

where $Q(p,\lambda)$ is a *quadratic matrix polynomial* of form

$$Q(p,\lambda) = \lambda^2 M(p) + \lambda C(p) + K(p), \tag{1.2}$$

$p = (p_1, p_2, \dots, p_m)^\top \in \mathbb{C}^m$ is a complex parameter vector, $M(p), C(p), K(p) \in \mathbb{C}^{n\times n}$ are analytic matrix valued functions of p on some domain $\mathcal{D}$ of $\mathbb{C}^m$. The scalar $\lambda \in \mathbb{C}$ and vectors $u, v \in \mathbb{C}^n$ satisfying (1.1) are called the *eigenvalue* and its corresponding *right and left eigenvectors* of QEP, respectively. It is obvious that λ, u and v are functions of p, i.e., $\lambda = \lambda(p)$, $u = u(p)$, $v = v(p)$. Here after we refer (λ, u, v) as an *eigen-triplet* of the QEP.

In this paper, we discuss the rounding errors of partial derivatives of simple eigenvalues of the QEP. The derivatives of the eigenvalues are widely used in many fields such as structural design optimization [1], model updating [2] and damage detection [3,4]. A large literature exists on computing derivatives of eigenvalues and eigenvectors in both theoretical and algorithmic aspects. Theoretically, in 1985, Sun [5] first used the implicit function theorem to prove the analyticity theorem of simple eigenvalue and its corresponding right and left eigenvectors of a matrix dependent on several parameters. In 1993, Andrew et al. [6] simplified and extended the discussion of Sun to nonlinear matrix functions. A lot of work has been done on numerical methods, such as Nelson's method [7,8], algebraic methods [9,10], modal methods [11,4] and recently the single mode method [12]. In actual practice, the true value of the derivative of the simple eigenvalue is usually unknown,

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so there is a problem that we cannot evaluate how accurately the numerical methods are able to compute the eigenvalue derivatives. If we can give a rounding error analysis on the eigenvalue derivative, this problem would be considered to be solved. However, we are not aware of any previously published paper working on rounding errors of partial derivatives of a simple eigenvalue of the QEP, so we focus on deriving rounding errors in this paper. As we know, most methods compute partial derivatives of a simple eigenvalue $\lambda(p)$ with respect to p_k at p_* based on the following equations (see [7,9,11,10,12] for details):

$$D(p_*) = 2\lambda(p_*)M(p_*) + C(p_*), \tag{1.3}$$

$$\sigma(p_*) = v(p_*)^\top D(p_*)u(p_*), \tag{1.4}$$

$$R(p_*) = \lambda(p_*)^2 \frac{\partial M(p_*)}{\partial p_k} + \lambda(p_*)\frac{\partial C(p_*)}{\partial p_k} + \frac{\partial K(p_*)}{\partial p_k}, \tag{1.5}$$

$$\frac{\partial \lambda(p_*)}{\partial p_k} = -\frac{v(p_*)^\top R(p_*)v(p_*)}{\sigma(p_*)}. \tag{1.6}$$

We derive rounding errors according to (1.3)–(1.6) and summarize the results in a theorem, and then we give another theorem for the symmetric QEP as a special case. At last, we give an example to verify the validity of our theorem, and numerical results show that our rounding error is a very good upper bound estimation of relative errors of partial derivatives of the simple eigenvalue.

Hereafter we suppose that there are no storing errors and we eliminate p_* in (1.3)–(1.6) for convenience, and we denote $\frac{\partial M}{\partial p_k}, \frac{\partial C}{\partial p_k}, \frac{\partial K}{\partial p_k}$ as $M_{p_k}, C_{p_k}, K_{p_k}$ for short respectively.

2. Rounding error

Assume that M, C, K and $M_{p_k}, C_{p_k}, K_{p_k}$ are all computed accurately and have already been stored in the computer. Denote the evaluation of an expression in floating point arithmetic as $\mathrm{fl}(\cdot)$, and let "$\circ$" represent the basic arithmetic operations $+, -, \times, /$. Let ε be the machine precision and assume that

$$1.01n\varepsilon \le 0.01 \tag{2.1}$$

throughout the rest of paper.

First we list four lemmas of rounding errors in [13,14].

Lemma 2.1. *Let α and β be real floating point numbers, then*

$$\mathrm{fl}(\alpha \circ \beta) = (\alpha \circ \beta)(1+\delta), \quad |\delta| \le \varepsilon.$$

Lemma 2.2. *If $|\delta_i| \le \varepsilon$ and $\rho_i = \pm 1$ for $i = 1, \dots, n$, and (2.1) holds, then*

$$\prod_{i=1}^{n}(1+\delta_i)^{\rho_i} = 1 + \theta_n,$$

where

$$|\theta_n| \le \frac{n\varepsilon}{1-n\varepsilon} =: \gamma_n \le 1.01n\varepsilon. \tag{2.2}$$

Lemmas 2.1 and 2.2 yield Lemma 2.3.

Lemma 2.3. *Let x, y be two vectors composed of n real floating point numbers, $x = (x_1, \dots, x_n)^\top$, $y = (y_1, \dots, y_n)^\top$. Then $\mathrm{fl}(x^\top y)$ satisfies*

$$|\mathrm{fl}(x^\top y) - x^\top y| \le \gamma_n |x|^\top |y| \le 1.01n\varepsilon |x|^\top |y|.$$

Let $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$ be complex floating point numbers (whose real and imaginary parts are both real floating point numbers), we compute

$$\begin{aligned} \alpha \pm \beta &= (\alpha_1 \pm \beta_1) + i(\alpha_2 \pm \beta_2), \\ \alpha\beta &= (\alpha_1\beta_1 - \alpha_2\beta_2) + i(\alpha_1\beta_2 + \alpha_2\beta_1), \\ \alpha/\beta &= \frac{\alpha_1\beta_1 + \alpha_2\beta_2}{\beta_1^2 + \beta_2^2} + i\frac{\alpha_2\beta_1 - \alpha_1\beta_2}{\beta_1^2 + \beta_2^2}. \end{aligned} \tag{2.3}$$