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Galerkin schemes and inverse boundary value problems in reflexive Banach spaces

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ABSTRACT

We develop the Galerkin method for a recent version of the Lax–Milgram theorem. The generation of the corresponding finite-dimensional subspaces for concrete boundary value problems leads us to consider certain biorthogonal systems in the reflexive Banach spaces in question. In addition, we present an application to the numerical solution of inverse problems involving certain elliptic boundary value problems.

1. Introduction

The celebrated Lax–Milgram theorem [1] is a fundamental tool in the theory of variational formulations of linear elliptic partial differential equations, as well as for their numerical solution. According to its classical and well-known formulation, it guarantees that any coercive, continuous and bilinear form a on a Hilbert space H with inner product $\langle\cdot\rangle$ represents the space, in the sense that for all $y_0 \in H$ there exists a unique $x_0 \in H$ such that

$$y \in H \Rightarrow a(x_0, y) = \langle y_0, y\rangle, \tag{1.1}$$

and in addition $\|x_0\|$ depends both on the coercivity constant of a and $\|y_0\|$. Recently, in [2] an extension of this result has been stated in the setting of locally convex spaces. Such a version of the Lax–Milgram theorem not only generalizes the framework in which this result works, but also characterizes when a fixed functional is represented by an adequate bilinear form. In the normed case this characterization is given in terms of the existence of a constant.

In this paper we deal with several questions. On one hand, we derive the numerical stability of the Galerkin method corresponding to the mentioned Lax–Milgram-type result. In connection with this kind of schemes for different problems,

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let us mention that an intensive research has been developed in the last decades, as show the recent works [3–11] and the references therein. On the other hand, we apply the generalized Lax–Milgram theorem to solve numerically some boundary value inverse problems, thus extending the applicability of the collage method proposed in [12,13].

More precisely, in Section 2 we recall the Lax–Milgram theorem [2, Theorem 1.2] for those locally convex spaces for which the notion of approximation makes sense, that is, for the normed ones, and we consider the corresponding Galerkin approximation scheme and analyze its numerical stability, arriving at a version of Céa's inequality in this context. In Section 3, the use of adequate biorthogonal systems in certain reflexive Banach spaces allows us to generate Galerkin's methods for the approximation of a wide class of problems, as well as stating their stability. We illustrate it with some numerical examples. Finally, in Section 4 we modify the collage type result given in [12,13] by means of the generalized Lax–Milgram theorem and the use of biorthogonal systems in reflexive Banach spaces. This generalization provides us with a numerical method for solving new examples of boundary value inverse problems for which the original collage theorem does not apply. We also include an example concerning the numerical solution of boundary value inverse problems. For simplicity, we deal only with real spaces, although our results can also be established for the complex case in an easy and straightforward way.

2. Numerical stability of the Galerkin scheme

In this section we recall the mentioned version of the Lax–Milgram theorem, but for the reflexive context, instead for the more general locally convex one, since it suffices for our purposes. We also provide the corresponding Galerkin approach to that generalized Lax–Milgram theorem, as well as an extension of the classical Céa estimate of the error.

We begin by reviewing some standard notation: given real linear spaces E and F, a bilinear form $a : E \times F \longrightarrow \mathbb{R},\ x_0 \in E$ and $y_0 \in F$, $a(\cdot, y_0)$ denotes the linear functional on E

$$x \in E \longmapsto a(x, y_0) \in \mathbb{R},$$

whereas $a(x_0, \cdot)$ stands for the analogous linear functional on F. Additionally, given a real normed space E, we write E^* for its topological dual space. Finally, $()_+$ is the positive part, i.e., for $t \in \mathbb{R},\ (t)_+ = \max\{t, 0\}$. The generalized Lax–Milgram theorem is stated in these terms (see [2, Corollary 1.3]):

Theorem 2.1. *Assume that E is a real reflexive Banach space and that F is a real normed space, $y_0 \in F^*,\ a : E \times F \longrightarrow \mathbb{R}$ is bilinear and that C is a nonempty convex subset of F such that for all $y \in C,\ a(\cdot, y) \in E^*$. Then*

$$\text{there exists } x_0 \in E \text{ such that for all } y \in C, \quad y_0^*(y) \le a(x_0, y) \tag{2.1}$$

if, and only if,

$$\text{there exists } \alpha > 0 \text{ such that for all } y \in C, \quad y_0^*(y) \le \alpha \|a(\cdot, y)\|. \tag{2.2}$$

In addition, if one of these equivalent statements is satisfied and for some $y \in C$ we have that $a(\cdot, y) \neq 0$, then

$$\min\{\|x_0\| : x_0 \in E \text{ and for all } y \in C,\ y_0^*(y) \le a(x_0, y)\} = \left(\sup_{y \in C,\ a(\cdot,y)\neq 0} \frac{y_0^*(y)}{\|a(\cdot, y)\|} \right)_+ . \tag{2.3}$$

Let us note that when C is balanced, that is, $C = -C$, inequality (2.1) becomes an equality. Furthermore, for such a C we have an easy but useful result characterizing the uniqueness in the variational inequality (2.1), which is nothing more than that of the corresponding homogeneous problem:

Lemma 2.2. *Let E and F be real vector spaces, let C be a nonempty convex and balanced subset of F, let $a : E \times F \longrightarrow \mathbb{R}$ be a bilinear form and let $y_0^* : F \longrightarrow \mathbb{R}$ be a linear functional. Suppose that the variational equation*

$$\text{find } x_0 \in X \text{ such that } y \in C \Rightarrow y_0^*(y) = a(x_0, y) \tag{2.4}$$

has a solution. Then, it is unique if, and only if,

$$x \in E \text{ and for all } y \in C, \quad a(x, y) = 0 \Rightarrow x = 0. \tag{2.5}$$

Proof. Assume that problem (2.4) admits a unique solution $x_0 \in E$, and let $x \in E$ such that

$$\text{for all } y \in C, \quad a(x, y) = 0.$$

Then

$$\text{for all } y \in C, \quad y_0^*(y) = a(x_0 + x, y)$$