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A two-grid decoupling method for the mixed Stokes–Darcy model[☆]

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ABSTRACT

In this paper, we consider the mixed Stokes–Darcy problem which describes a fluid flow coupled with a porous media. We present a modified two-grid method for decoupling this mixed model. Stability is proved and optimal error estimates are derived. The numerical results show that the modified two-grid method is effective and has the same accuracy as the coupling scheme when we choose $h = H^2$.

1. Introduction

Multi-modeling problems have received a wide publicity over the past years due to their applications. We mainly have interest in the mixed Stokes–Darcy model, which describes the coupling of the fluid flow (governed by the Stokes equations) with the porous media flow (governed by the Darcy equations) through certain interface conditions.

Generally speaking, solving this coupled model usually results in difficulties, especially in the numerical implementation. We are interested in the decoupling methods, in which the coupled problems can be separated into two single flow subproblems. This allows us to use suitable methods flexibly for solving each subproblem separately, and numerical implementation is easy and efficient.

Some decoupling technologies have been developed during the last decades. For example, the domain decomposition methods for various coupled models were studied in [1–12], the Lagrange multiplier approach is used in [13,14], the interface relaxation approach is applied in [15,16], precondition techniques are considered in [17,18], and the decoupling marching algorithm based on interface approximation via temporal extrapolation is proposed in [19–22] for same time step length in both subproblems or different time step lengths in different subproblems. In addition, the two-grid method is applied successfully to solve multi-modeling problems in [23–25], using this method, one can get two independent subproblems on certain fine grid since the transmission conditions on the interface can be approximated by the coarse grid approximation. Furthermore, a multilevel decoupled method for the mixed Stokes–Darcy model is proposed in [26]. However, such two-grid and multilevel schemes have not theoretically achieved the optimal error estimates about the L^2-norm of ∇u and p since the interface condition is approximated by the coarse grid approximation. In this paper, we will follow the idea in [25] and propose a modified two-grid method for the mixed Stokes–Darcy model with the Beavers–Joseph–Saffman interface condition. The optimal error estimates are obtained for this method.

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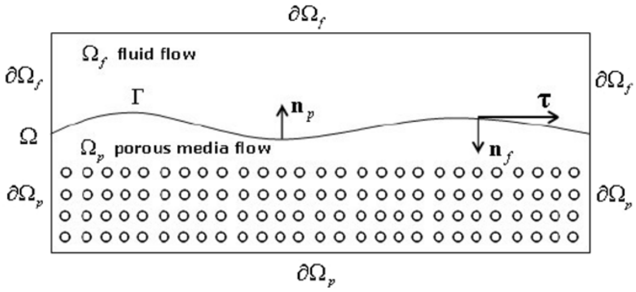

Fig. 1. A global domain Ω consisting of a fluid region Ω_f and a porous region Ω_p separated by an interface Γ.

The rest of the paper is organized as follows. A mixed Stokes–Darcy model is described in the next section. In Section 3, the modified two-grid algorithm is given. Stability is proved and convergence is deduced in Section 4 followed by some numerical experiments in Section 5. Finally, we end this paper with a conclusion in Section 6.

2. Model problem

We consider the model in a bounded domain $\Omega \subset R^d$ ($d = 2$ or 3), which consists of a fluid flow region Ω_f and a porous media region Ω_p, see Fig. 1. Here $\Omega_f \cap \Omega_p = \emptyset$, $\overline{\Omega}_f \cup \overline{\Omega}_p = \overline{\Omega}$. Two domains are separated by the interface $\Gamma = \partial\Omega_f \cap \partial\Omega_p$. Let $\Gamma_f = \partial\Omega_f \setminus \Gamma,\ \Gamma_p = \partial\Omega_p \setminus \Gamma$,

The Stokes equations govern the fluid flow in Ω_f:

$$-\nu\Delta u + \nabla p = f_1 \quad \text{in } \Omega_f, \tag{2.1}$$

$$\nabla \cdot u = 0 \quad \text{in } \Omega_f. \tag{2.2}$$

Here, u and p are the velocity field and the kinetic pressure, ν is the kinetic viscosity, f_1 is the external force.

The Darcy equations govern the porous media flow in Ω_p:

$$u_p = -\mathbf{K}\nabla\varphi \quad \text{in } \Omega_p, \tag{2.3}$$

$$\nabla \cdot u_p = f_2 \quad \text{in } \Omega_p. \tag{2.4}$$

Here u_p and φ are the Darcy velocity and the piezometric head, $\mathbf{K} = \{K_{ij}\}_{d\times d}$ is hydraulic conductivity tensor, denoting permeability of the rock. In this paper, we assume $\mathbf{K}$ is a symmetric and positive definite matrix with the smallest eigenvalue $k_{\min} > 0$. f_2 is the source term satisfying the solvability condition

$$\int_{\Omega_p} f_2 = 0.$$

Combining Darcy's law (2.3) with the continuity equation (2.4), we get the following elliptic equation:

$$-\nabla \cdot (\mathbf{K}\nabla\varphi) = f_2 \quad \text{in } \Omega_p. \tag{2.5}$$

On the interface Γ, we impose the following three interface conditions:

$$u \cdot n_f + u_p \cdot n_p = 0 \quad \text{on } \Gamma, \tag{2.6}$$

$$p - \nu n_f \frac{\partial u}{\partial n_f} = g\varphi \quad \text{on } \Gamma, \tag{2.7}$$

$$-\nu\tau_i \frac{\partial u}{\partial n_f} = \alpha\sqrt{\frac{\nu g}{\mathrm{tr}(\mathbf{K})}}\, u \cdot \tau_i \quad 1 \le i \le (d-1),\ \text{on } \Gamma, \tag{2.8}$$

where, n_f and n_p denote the unit outward normal vectors on $\partial\Omega_f$ and $\partial\Omega_p$, in particularly, $n_p = -n_f$ on the interface Γ. g is the gravitational acceleration, α is a positive parameter depending on the properties of the porous medium, $\tau_i,\ i = 1, \ldots, d-1$, are the unit tangential vectors on the interface Γ. (2.8) is referred to as the Beavers–Joseph–Saffman interface condition, which is the simplified Beavers–Joseph interface condition.

For simplicity, we assume homogeneous Dirichlet boundary conditions are satisfied on Γ_f and Γ_p:

$$u = 0 \quad \text{on } \Gamma_f, \tag{2.9}$$

$$\varphi = 0 \quad \text{on } \Gamma_p. \tag{2.10}$$

The exact boundary conditions chosen above are not essential to either the analysis or the algorithms studied herein.