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Stability of numerical methods for jump diffusions and Markovian switching jump diffusions

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ABSTRACT

This work focuses on stability analysis of numerical solutions to jump diffusions and jump diffusions with Markovian switching. Due to the use of Poisson processes, using asymptotic expansions as in the usual approach of treating diffusion processes does not work. Different from the existing treatments of Euler–Maruyama methods for solutions of stochastic differential equations, we use techniques from stochastic approximation. We analyze the almost sure exponential stability and exponential p-stability. The benchmark test model in numerical solutions, namely, one-dimensional linear scalar jump diffusion is examined first and easily verifiable conditions are presented. Then Markovian regime-switching jump diffusions are dealt with. Moreover, analysis of stability of numerical methods for linearizable and multi-dimensional jump diffusions is carried out.

1. Introduction

This work focuses on stability of numerical solutions to jump diffusions and regime-switching jump diffusions. Since systems often run for an extended time, stability is of critical importance. As a result, much effort has been devoted to the stability analysis in the literature; see [1] for stability of diffusion processes, [2] for Markovian switching diffusions, and [3] for switching diffusions in which the switching depends on the diffusion parts. In practice, closed-form solutions are difficult to obtain. Numerical methods are more viable or even the only possible alternative to solve the problems. Starting with a specific problem, an immediate question is: If the system of interest is stable, what can be said about the corresponding numerical approximation?

Because of the importance, there has been much work on numerics of diffusions, jump diffusions, and their regime-switching counterparts. General survey and classical treatments can be found in [4,5] and references therein. In [6], almost sure exponential stability of the Euler–Maruyama (E–M) algorithm as well as that of exponential p-stability was treated for diffusion systems. In [7], almost sure exponential stability and exponential p-stability for the E–M algorithm were studied for Markovian switching diffusions. In [8], asymptotical stability in the large of E–M algorithm was examined for jump diffusion systems. In [9], mean square stability and asymptotical stability in the large of stochastic theta method were presented. In [10], the split-step backward Euler method and compensated split-step backward Euler method were analyzed and strong convergence results were obtained under certain assumptions for nonlinear jump diffusion systems. In [2], mean square stability was treated for Markovian regime-switching diffusions. In lieu of the Brownian increments, i.i.d. sequences were used

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and pathwise convergence rates for diffusions were dealt with in [11] by consideration of re-embedded sequences. Given the key roles of jump-diffusions played in networked systems, this paper is devoted to answering stability questions of numerical solutions to jump diffusions. Although there have been many excellent works on numerical solutions of stochastic differential equations, the study on numerical methods of almost sure exponential stability and exponential p-stability for jump diffusions has not been done yet to the best of our knowledge. One intuitive thought might be: Perhaps one can repeat the success in the numerical approximation to diffusions, in which the techniques used were based on asymptotic series of expansion of moments of Brownian motion. A scrutiny, however, shows that such an approach is not going to work. This is because a Gaussian distribution is completely determined by the first and second moments, whereas for a Poisson random variable, the mean and variance are the same. Thus using expansion of the Poisson increments will not produce higher powers in terms of the small step size in contrast to the Brownian increments. This rules out the possibility of using existing techniques in the current setup. To illustrate, let us start with an algorithm with step size $\varepsilon > 0$. The increment of a standard Brownian motion $\Delta w = w(\varepsilon(n+1)) - w(\varepsilon n)$ satisfies

$$E\Delta w = 0, \qquad E(\Delta w)^2 = \varepsilon, \qquad E(\Delta w)^{2n} = \frac{(2n)!}{n!2^n}\varepsilon^n$$

and all odd moments of increment of Brownian motion are 0. Thus it is advantageous to use series expansions since the higher the moment, the higher order of ε. In contrast, unlike the increments of Brownian motion, the increments of a Poisson process behave very differently. In fact, since $\Delta N \sim$Poisson$(\lambda\varepsilon)$, $E\Delta N = \lambda\varepsilon$, $\mathrm{Var}(\Delta N) = \lambda\varepsilon$, and $E(\Delta N)^2 = \lambda\varepsilon + (\lambda\varepsilon)^2$, and

$$E(\Delta N)^n = \sum_{i=1}^{n} (\lambda\varepsilon)^i \frac{1}{i!} \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} j^n.$$

A moment of reflection reveals that in the nth moment of ΔN, the leading term is $\lambda\varepsilon$ for all n. That is, higher moments do not yield higher order of ε (in terms of order of magnitude estimates). Therefore, using series expansion methods appears not applicable.

Focusing on stability, our contributions in this paper include the following aspects. (1) We obtain sufficient conditions for pth-moment stability and almost sure stability for jump diffusions and jump diffusions with regime switchings. (2) We provide conditions under which the stability of the stochastic differential equations (in a certain sense) implies stability of the associated numerical algorithms. Thus, we answer the question: Passing from the original systems to that of the numerical solutions, under what conditions, stability will be preserved. (3) In the traditional approach for numerical methods of stochastic differential equations, one often has to use Taylor expansions. For Poisson processes, since the mean and variance are the same, the Taylor expansions do not really help. We use techniques from stochastic approximation toolbox, which enables us to resolve the problem and obtain convergence and stability. Using our definitions of stability for the numerical algorithms, stability of numerical algorithms will imply that of SDEs. Not only are these questions important from a theoretical point of view, but also they provide crucial practical insight for actual computing. To get the insight and to make comparisons, we first begin with one-dimensional benchmark models. We then further our study for considering multi-dimensional cases and systems with switching.

The rest of the paper is arranged as follows. For the reason of better visualization and presentation, we begin with a simple benchmark model in Section 2. The problem is a simple linear scalar equation, whose closed-form solution is readily available. Thus comparisons of the closed-form solution and numerical approximation can be made relatively easily. Our emphases are on the analysis of almost sure exponential stability and exponential p-stability of the algorithms. Also examined are the Markovian switching jump diffusion counterparts. Section 3 furthers our investigation by generalizing the model to non-linear multi-dimensional systems. Section 4 presents a couple examples to demonstrate our results. Finally, the paper is concluded in Section 5 with further remarks.

2. Benchmark model and algorithm

This section focuses on a benchmark model, namely, a one-dimensional linear scalar jump diffusion equation. We let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete filtered probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual condition (i.e., it is right continuous with $\mathcal{F}_0$ containing all P-null sets). Consider the benchmark test model

$$\begin{aligned} &dX(t) = bX(t)dt + \sigma X(t)dw(t) + \gamma X(t^-)dN(t) \\ &X(0) = x, \end{aligned} \tag{2.1}$$

where b, σ, and γ are real constants, $w(t)$ is a scalar Brownian motion, and $N(t)$ is a scalar Poisson process independent of the Brownian motion. In what follows, we write the solution of (2.1) as $X^x(t)$ to emphasize its initial data x dependence. It is easy to see that 0 is the only equilibrium point of the dynamic system. Define a compensated or centered Poisson process as

$$\widetilde{N}(t) = N(t) - \lambda t, \tag{2.2}$$

where $0 < \lambda < \infty$ is known as the jump rate. To proceed, we first recall the following definitions of stability, which were originated from the diffusion counterparts in [1].