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Numerical approximation of distributed order reaction–diffusion equations

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ABSTRACT

In this paper an implicit scheme for the numerical approximation of the distributed order time-fractional reaction–diffusion equation with a nonlinear source term is presented. The stability and the convergence order of the numerical scheme are analysed and illustrated through some numerical examples.

1. Introduction

In the recent decades a considerable and increasing attention has been devoted to fractional differential equations mainly because it has been observed that models including noninteger orders in their derivatives could describe accurately certain processes than those that restrict the orders of the derivatives to integer values. Application problems of fractional differential equations are nowadays recognized in several areas of science and engineering (see, for example book [1], that contains a description of the several applications of Fractional Calculus, as well as a survey of numerical methods for fractional differential equations). The time-fractional reaction–diffusion–wave equation (TFDWE)

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = k(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t,u(x,t)), \quad t > 0,\ 0 \le x \le L, \tag{1}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ represents a fractional derivative with order α, $0 < \alpha \le 2$, has been found in a broad variety of engineering, biological and physics processes where anomalous diffusion occurs [2–7] such as the ones occurring in sub-diffusive or

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super-diffusive processes. When $0 < \alpha < 1$, Eq. (1) is a time-fractional diffusion equation, while if $1 < \alpha < 2$, it is a time-fractional wave–diffusion equation. In the case where $\alpha = 1$, we obtain the classical diffusion equation and when $\alpha = 2$ we obtain the classical wave equation.

Concerning the numerical approximation of such equations, finite difference methods are the most commonly find (see for example [8–19]) although other numerical approaches have also been used, as for example, finite element methods [20,21], meshless collocation methods [22], collocation spectral methods [23], nonpolynomial collocation schemes [24].

Recently, some attention has also been devoted to a more general class of problems than (1), by considering the so-called distributed order equations. The reaction–wave–diffusion equation with distributed order in time may be written as:

$$\int_a^b c(\alpha)\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\,d\alpha = k(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t,u(x,t)), \quad t > 0,\ 0 \le x \le L, \tag{2}$$

$0 \le a < b \le 2$, where the fractional derivative is given in the Caputo sense (see [25]):

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{\partial^n u(x,s)}{\partial s^n}\,ds, \tag{3}$$

where n is the smallest integer greater than or equal to α, and the function $c(\alpha)$ acting as weight for the order of differentiation is such that [26,27] $c(\alpha) > 0$ and $\int_a^b c(\alpha)\,d\alpha = C > 0$. If $a = 0$ and $b = 1$ we obtain the diffusion equation with distributed order in time, and if $a = 1$ and $b = 2$, we have the wave–diffusion equation with distributed order in time. In [28,29], both time and space distributed order diffusion equations were investigated. In [30], the diffusion equation of distributed order in time (between zero and one) was analysed for Dirichlet, Neumann and Cauchy boundary conditions. The physical justification and some analytical properties of the time-fractional diffusion equation of distributed order may also be found in [26,31], [27] and the references therein. As far as we are aware, no numerical methods have been reported yet for the numerical approximation of (2). A numerical method for the diffusion equation without a reaction term, with distributed order in time (over the interval [0, 1]), has been presented very recently in [32,33]. Due to some application problems not included in the problems investigated in these two recent works, as for example, problems that are commonly used to model the growth and spreading of biological species [34–36], or to model the electrical activity in a neuron (the FitzHugh–Nagumo equation [37,38]), we have decided to extend the results obtained there to the case where we consider (2) with a nonlinear source term, $a = 0,\ b = 1$, with initial condition:

$$u(x,0) = g(x), \quad 0 < x < L \tag{4}$$

and the boundary conditions:

$$u(0,t) = \varphi_0(t), \qquad u(L,t) = \varphi_L(t), \quad t > 0 \tag{5}$$

where we assume that $g, f, \varphi_0, \varphi_L, k$ and c are continuous, k is nonnegative and the fractional derivative is given in the Caputo sense.

We also assume that the nonlinear source term f satisfies a Lipschitz condition of the form

$$|f(x,t,u) - f(x,t,v)| \le L_f\,|u - v|\,, \tag{6}$$

where L_f is a positive constant.

The paper is organized in the following way: in Section 2 we describe the method to approximate the solution of (2), (4), (5). In Section 3 we analyse the stability and the convergence of the numerical scheme. In Section 4 we present and discuss some numerical examples and we end with some conclusions and plans for further investigation.

2. The numerical method

In this section we present an implicit numerical scheme for the approximation of the solution of (2), (4), (5) with $a = 0, b = 1$.

First, we use a quadrature rule to approximate the integral term on the left-hand side of (2). We consider a partition of the interval $[a, b]$, the interval where the order of the time derivative lies, into N subintervals, $[\beta_{j-1}, \beta_j],\ j = 1, \ldots, N$, of equal amplitude $h = (b-a)/N$. Denoting the midpoints of each one of these subintervals, by

$$\alpha_j = \frac{\beta_{j-1} + \beta_j}{2}, \quad j = 1, \ldots, N,$$

we use the midpoint rule to approximate the integral term in (2), obtaining

$$\int_a^b c(\alpha)\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\,d\alpha = h\sum_{j=1}^{N} c(\alpha_j)\frac{\partial^{\alpha_j} u(x,t)}{\partial t^{\alpha_j}} - \frac{h^2}{24}H''(\nu), \quad \nu \in (a,b), \tag{7}$$

where H is defined by:

$$H(\alpha) = c(\alpha)\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}. \tag{8}$$