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The application of Newton's method in vector form for solving nonlinear scalar equations where the classical Newton method fails

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ABSTRACT

In this paper we propose a strategy to obtain the solutions of a scalar equation $f(x) = 0$ through solving an associated system of two equations. In this way, the solutions of the associated system lying on the identity line provide solutions of the given equation. In most cases, the associated system cannot be solved exactly. Solving this system with the Newton method for systems may result more efficient than the application of the scalar Newton method to the simpler equation. In some pathological cases in which the scalar Newton method does not provide solution, this strategy works appropriately. Some examples are given to illustrate the performance of the proposed strategy.

1. Introduction

Finding the roots of a nonlinear equation $f(x) = 0$ with $f(x) : X \to X,\ X \subset \mathbb{R}$, is a classical and important problem in science and engineering. There are very few functions for which the roots can be expressed explicitly in closed form. Thus, the solutions must be obtained approximately, relying on numerical techniques based on iterative processes [1]. Given an initial guess for the root, x_0, successive approximations are obtained by means of an iteration function (IF) $\Phi : X \longrightarrow X$

$$x_{n+1} = \Phi(x_n), \quad n = 0, 1, 2 \ldots$$

which will often converge to a root α of the equation, provided that some convergence criterion is satisfied.

If there exists a real number $p \geq 1$ and a nonzero constant $A > 0$ such that

$$\lim_{n\to\infty} \frac{|\Phi(x_n) - \alpha|}{|x_n - \alpha|^p} = A$$

then p is called the order of convergence, and A is the asymptotic error constant of the method defined by Φ. For such a method, the error $|e_{n+1}| = |x_{n+1} - \alpha|$ is proportional to $|e_n|^p = |x_n - \alpha|^p$ as $n \to \infty$. In practice, the order of convergence is often determined by using Schröder–Traub's theorem [2] which states that being Φ an IF with $\Phi^{(r)}$ continuous in a neighborhood of α, then Φ is of order r if and only if

$$\Phi(\alpha) = \alpha, \qquad \Phi'(\alpha) = \cdots = \Phi^{(r-1)}(\alpha) = 0, \qquad \Phi^{(r)}(\alpha) \neq 0.$$

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Given an IF Φ, the orbit of $x_0 \in X$ under Φ is defined as the set given by the sequence of points

$$\mathrm{orb}(x_0) = \left\{x_0, \Phi(x_0), \ldots, \Phi^k(x_0), \ldots\right\}$$

where $\Phi^k(x)$ denotes the k-fold composition of Φ with itself applied to x_0, that is, $\Phi^k(x_0) = \Phi(\Phi(\ldots\Phi(x_0)\ldots))$ where Φ is applied k times. The point x_0 is called the seed of the orbit. A point x_0 is a fixed or equilibrium point of Φ if $\Phi(x_0) = x_0$. Note that the orbit of a fixed point x_0 consists on the constant sequence $\{x_0\}$.

If $\Phi^n(x_0) = x_0$ and $\Phi^j(x_0) \neq x_0$ for $0 < j < n$ then x_0 is called a point with period n. In this case the corresponding orbit is said to be a periodic orbit of period n, or a n-cycle. All points on the orbit of x_0 have the same period. Note that if x_0 is a periodic point of period $n \geq 1$ then x_0 is a fixed point of Φ^n, and the corresponding orbit has only n different points. Similarly, if x_0 is a periodic point of period $n \geq 1$ then x_0 is also fixed by Φ^{kn} for any positive integer k.

A point x_0 is called eventually fixed or eventually periodic if x_0 itself is not fixed or periodic, but some point on $\mathrm{orb}(x_0)$ is fixed or periodic.

A fixed point x_0 of Φ is attracting if $|\Phi'(x_0)| < 1$, is repelling if $|\Phi'(x_0)| > 1$, and is indifferent or neutral if $|\Phi'(x_0)| = 1$. Readers interested in the above concepts, belonging to the topic of dynamical systems, may see Refs. [3–5].

2. The Newton–Raphson method

We will consider that the function $f(x)$ whose root need to be calculated is a scalar real one, $f : \mathbb{R} \to \mathbb{R}$. Among the iteration methods, the Newton method is probably the best known and most reliable and most used algorithm [6,7]. There are many ways to obtain Newton method. We will offer a brief outline of the geometric approach. The basic idea behind Newton's method relies in approximating $f(x)$ with the best linear approximation passing for an initial guess x_0 which is near the root α. This best linear approximation is the tangent line to the graph of $f(x)$ at $(x_0, f(x_0))$. Provided $f'(x_0) \neq 0$, the intersection of this line with the horizontal axis results in

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

This process can be continued with a new approximation x_1 to get x_2, and so on. In this way, the Newton iteration function is given by

$$N_f(x) = x - \frac{f(x)}{f'(x)}.$$

It converges quadratically to simple zeros and linearly to multiple zeros, provided a good approximation is at hand. There exist different results about semilocal convergence of Newton's method in which precise bounds are given for balls of convergence and uniqueness. Nevertheless, the dynamics of the Newton IF N_f may be very complicated, even for apparently simple functions. We expect that the sequence obtained through the IF will converge to the desired root of $f(x)$. Nevertheless, depending on the function under consideration and the initial point selected, the orbit of this point may not always converge successfully. Some pathological situations in the application of Newton's method follow:

1. Newton's method may fail if at any stage of computation the derivative of the function $f(x)$ is either zero or very small. For low values of the derivative, the Newton iteration offshoots away from the current point of iteration and may possibly converge to a root far away from the desired one.
2. It may fail also to converge to the desired root in case the initial point is far from this root.
3. For certain forms of equations, Newton method diverges. A typical example of this situation is the function $f(x) = \sqrt[3]{x}$, for which the iteration function is $N_f(x) = 2x$, which results in a divergent sequence whatever the initial guess $x_0 \neq 0$ is.
4. In some cases, the iterations oscillate. For example, this is the situation if the initial guess is in an orbit of period n or is eventually periodic of period n.
5. The convergence of Newton's method may be very slow near roots of multiplicity greater than one. In fact, for multiple roots the Newton method looses its quadratic convergence.
6. Finally, the most dramatic situation occurs when the iteration function N_f exhibits chaotic behavior, in which there are orbits that wander from here to there aimlessly.

An important result to be considered here is the following theorem whose proof may be found in [3, p. 167].

Theorem 1 (*Newton's Fixed Point Theorem*)**.** *α is a root of $f(x)$ of multiplicity k if and only if α is a fixed point of the IF N_f. Moreover, such a fixed point is always attracting.*

The importance of this result resides in the fact that the iteration function N_f does not have extraneous fixed points (which are not solutions of $f(x) = 0$). These extraneous fixed points appear in other iteration functions of well-known methods such as the Halley method or the Chebyshev method.

3. The associated system to the scalar equation

Let us consider a particularization of a result in [8].

Proposition 1. *Let be $f(x) : \mathbb{R} \to \mathbb{R}$ a differentiable function, and let $\mathcal{O} = \{x_1, x_2\} \in \mathbb{R}$ with $x_1 \neq x_2$. Then $\mathcal{O}$ is a 2-cycle of the Newton IF N_f if and only if $x = x_1, y = x_2$ is a solution of the system $\left\{N_f(x) = y, N_f(y) = x\right\}$.*