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Almost strictly totally negative matrices: An algorithmic characterization

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ABSTRACT

A real matrix $A = (a_{ij})_{1\leq i,j,\leq n}$ is said to be almost strictly totally negative if it is almost strictly sign regular with signature $\varepsilon = (-1, -1, \ldots, -1)$, which is equivalent to the property that all its nontrivial minors are negative. In this paper an algorithmic characterization of nonsingular almost strictly totally negative matrices is presented.

1. Introduction

Matrices with all its minors nonnegative, known as Totally Positive (TP) matrices and also as totally nonnegative matrices, have been widely studied since mid of the last century (see e.g. [1–3]). They form a subclass of the set of the Sign Regular (SR) matrices, whose minors of the same order have the same sign (see e.g. [1,4]). In general, problems with SR matrices are more difficult than the corresponding ones with TP matrices. SR matrices also contain the class of Totally Negative (TN) matrices, also called totally nonpositive matrices, which have been recently studied (see [5–8]). In [8] Strictly Totally Negative (STN) matrices are defined as those with all their minors negative.

Among the SR matrices, an important particular subclass is that of the Almost Strictly Sign Regular (ASSR) matrices, defined by R. Huang et al. in [9] as those whose nontrivial minors of the same order have all the same strict sign. It contains the class of Almost Strictly Totally Positive (ASTP) matrices (see [10]). ASSR matrices were characterized through the Neville Elimination (NE) in [11]. In this work the authors deal with another subset of ASSR matrices called Almost Strictly Totally Negative (ASTN). All nontrivial minor of these matrices are strictly negative, which notably simplifies the characterization proposed in [11] for ASSR matrices. ASTN matrices form an intermediate class of matrices between TN and STN matrices. The goal of our paper is an algorithmic characterization of ASTN matrices through NE.

The NE is an alternative procedure to Gaussian elimination for reducing a square matrix to upper triangular form, preferable for some classes of matrices and when using pivoting strategies in parallel implementations. Roughly speaking, the Neville elimination introduces zeros in each column of a matrix by adding to each row an appropriate multiple of the previous one (instead of using a single row with a fixed pivot, as in Gaussian elimination).

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If A is an $n \times n$ matrix, it consists of at most $n-1$ successive major steps, resulting in a sequence of matrices as follows:

$$A = \widetilde{A}^{(1)} \to A^{(1)} \to \widetilde{A}^{(2)} \to A^{(2)} \to \cdots \to \widetilde{A}^{(n)} = A^{(n)} = U, \tag{1}$$

where U is an upper triangular matrix.

For each t, $1 \le t \le$, both $A^{(t)}$ and $\widetilde{A}^{(t)}$ have zeros below their diagonal entries in the first $t-1$ columns; furthermore

$$a_{it}^{(t)} = 0, \quad i \ge t \Rightarrow a_{ht}^{(t)} = 0, \ \forall\, h \ge i. \tag{2}$$

Matrix $A^{(t)}$ is obtained from $\widetilde{A}^{(t)}$ reordering rows $t, t+1, \ldots, n$ according to a row pivoting strategy which satisfies (2). To obtain $\widetilde{A}^{(t+1)}$ from $A^{(t)}$, zeros are introduced below the main diagonal of the tth column by subtracting a multiple of the ith row from the $(i+1)$th, for $i = n-1, n-1, \ldots, t$, according to the following formula

$$\widetilde{a}_{ij}^{(t+1)} = \begin{cases} a_{ij}^{(t)}, & \text{if } 1 \le i \le t, \\ a_{ij}^{(t)} - \dfrac{a_{it}^{(t)}}{a_{i-1,t}^{(t)}} a_{i-1,j}^{(t)}, & \text{if } t+1 \le i \le n \text{ and } a_{i-1,t}^{(t)} \ne 0, \\ a_{ij}^{(t)}, & \text{if } t+1 \le i \le n \text{ and } a_{i-1,t}^{(t)} = 0, \end{cases} \tag{3}$$

for all $j \in \{1, 2, \ldots, n\}$.

If A is nonsingular, the matrix $\widetilde{A}^{(t)}$ has zeros below its main diagonal in the first $t-1$ columns. Let us notice that in this process one has $\widetilde{A}^{(n)} = A^{(n)} = U$, and that when no rows exchanges are needed, then $\widetilde{A}^{(t)} = A^{(t)}$ for all t. The element

$$p_{ij} = a_{ij}^{(j)}, \quad 1 \le j \le i \le n, \tag{4}$$

is called the (i, j) pivot of NE of A and the number

$$m_{ij} = \begin{cases} \dfrac{a_{ij}^{(j)}}{a_{i-1,j}^{(j)}} = \dfrac{p_{ij}}{p_{i-1,j}}, & \text{if } a_{i-1,j}^{(j)} \ne 0, \\ 0, & \text{if } a_{i-1,j}^{(j)} = 0 \ (\Rightarrow a_{ij}^{(j)} = 0), \end{cases} \tag{5}$$

the (i, j) multiplier. Note that $m_{ij} = 0$ if and only if $p_{ij} = 0$ and by (2)

$$m_{ij} = 0 \Rightarrow m_{hj} = 0, \quad \forall\, h > i. \tag{6}$$

Furthermore, the NE is an efficient tool that allows us to obtain algorithms with high relative accuracy (HRA) for the computation of singular values, eigenvalues, inverses or the LDU factorization of certain kind of matrices. This is the case of matrices that admit a bidiagonal decomposition (see [12–14]). HRA means that the relative errors of the computations are of the order of machine precision, independently of the size of the condition number of matrix.

On the other hand, recently, some results for nonsingular totally negative (nonpositive) matrices are given (see [5–7]). However, up to our best knowledge, there is not any characterization of the ASTN matrices, which is the objective of this work.

The paper is organized as follows. Firstly basic notations are introduced, next some definitions and basic results on ASSR matrices are presented. Section 4 describes the definitions and results that allow us characterizing the ASTN matrices using the NE. Taking into account these results, it is possible to analyze if a matrix is ASTN using the algorithms developed at the end of the paper.

2. Basic notations

Let us start by introducing some classic notations. For $m, n \in \mathbb{N}$, with $1 \le m \le n$, $Q_{m,n}$ denotes the set of all increasing sequences of m natural numbers not greater than n. For $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\beta = (\beta_1, \ldots, \beta_m) \in Q_{m,n}$ and A an $n \times n$ real matrix, we denote by $A[\alpha|\beta]$ the $m \times m$ submatrix of A containing rows $\alpha_1, \ldots, \alpha_m$ and columns $\beta_1, \ldots, \beta_m$ of A. If $\alpha = \beta$, we denote by $A[\alpha] := A[\alpha|\alpha]$ the corresponding principal minor. $Q_{m,n}^0$ denotes the set of increasing sequences of m consecutive natural numbers not greater than n.

Many of matrices present grouped null elements in certain positions, in particular this occurs with the ASSR matrices, which can be classified in two classes that are defined below, type-I and type-II staircases.

A matrix $A = (a_{ij})_{1 \le i,j \le n}$ is called type-I staircase if it satisfies simultaneously the following conditions

- $a_{11} \ne 0, \ a_{22} \ne 0, \ldots, a_{nn} \ne 0;$
- $a_{ij} = 0, i > j \Rightarrow a_{kl} = 0, \forall l \le j, \ i \le k;$
- $a_{ij} = 0, i < j \Rightarrow a_{kl} = 0, \forall k \le i, \ j \le l.$

From now on it will be frequently used the backward identity matrix $n \times n$, P_n, whose element (i, j) is defined as

$$\begin{cases} 1, & \text{if } i + j = n + 1, \\ 0, & \text{otherwise.} \end{cases}$$

So, A is a type-II staircase matrix if it verifies that $P_n A$ is a type-I staircase matrix.