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On spline-based differential quadrature

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ABSTRACT

In the paper Barrera et al. (2014), a boolean sum differential quadrature method (DQM) was proposed by combining a spline interpolation operator having a fundamental function with minimal compact support and a spline quasi-interpolation operator reproducing the polynomials in the spline space. It is a general framework that provides results that differ from the ones obtained by defining specific schemes with structures which depend on the degree of the B-spline to be considered. The main drawback of these boolean sum DQMs is that the number of evaluation points increases quickly with the degree of the B-spline due to the use of a quasi-interpolation operator. We propose a different construction avoiding this problem and derive explicit results for low degree B-splines.

1. Introduction

The Differential Quadrature Method (DQM) is a numerical technique proposed by Bellman and coworkers in the early 1970s for solving differential equations by discretizing spatial derivatives by means of weighted sums of function values (cf. [1,2] and references quoted therein), i.e.

$$f^{(r)}(x_i) \simeq \sum_{j=1}^{N} a_{ij}^{(r)} f(x_j),$$

for some positive integer N and real numbers $a_{ij}^{(r)}$. Being the main task to determine the weights $a_{ij}^{(r)}$, Lagrangian polynomial interpolation is commonly used, so that the classical DQM is polynomial-based. It is well known that the number of grid points involved must be small to avoid the obtention of unstable solutions (see [3,4]). Therefore, some spline based DQMs have been proposed to overcome this drawback. Given a B-spline (cf. [5,6]), a cardinal Lagrangian or Hermitian interpolation spline with a compactly supported fundamental function is defined, from which the approximation of the derivatives is obtained. But the construction of this spline interpolant depends strongly on the degree of the B-spline (see for instance [7–10]). In [3], natural interpolating splines of odd degree are used to produce the spline interpolant from which the derivatives of the function to be approximated are computed. Also two algorithms are given in [11] to determine the weights $a_i^{(r)}$, although it is necessary to solve some intermediate linear systems of equations.

In [12], a boolean sum based DQM was proposed by combining a spline interpolation operator having a fundamental function with minimal compact support and a spline quasi-interpolation operator exact on the space of polynomials in the spline space. It is a general framework that provides results that differ from the ones obtained by defining specific schemes

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with a structure which depends on the degree of the B-spline to be considered. The main drawback of these boolean sum based DQMs is that the number of evaluation points increases quickly with the degree of the B-spline due to the use of a quasi-interpolation operator.

The goal of this paper is to improve the results obtained in [12] by providing a general method to construct DQMs from the solution of suitable spline interpolation problems. This method does not require the use of quasi-interpolation.

The remainder of this paper is structured as follows. In Section 2, we recall the two-stage construction based on quasi-interpolation. In Section 3, we consider the construction of DQMs without using quasi-interpolation. We give explicit results for low degree B-splines of even and odd orders in Sections 4 and 5, respectively. Finally, in Section 6 we analyze their corresponding error estimates.

2. Spline differential quadrature method based on quasi-interpolation

In [12], a general spline-based DQM was proposed by combining interpolation and quasi-interpolation. Let M_n be the B-spline of order $n \geq 2$ centered at the origin (cf. [5,13]). It is well known that it can be defined by successive convolutions: if

$$M_1(x) := \begin{cases} 1, & \text{if } x \in \left[-\frac{1}{2}, \frac{1}{2}\right), \\ 0, & \text{otherwise,} \end{cases}$$

is the characteristic function of the interval $\left[-\frac{1}{2}, \frac{1}{2}\right)$, then M_n is defined by the recurrence relation

$$M_n(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} M_{n-1}(t)\,dt, \quad n > 2.$$

It holds that M_n is a $C^{n-2}(\mathbb{R})$ piecewise polynomial function of degree $n-1$ supported on $\left[-\frac{n}{2}, \frac{n}{2}\right]$ having knots at the half-integers $\frac{1}{2} + \mathbb{Z}$ (resp. at the integers $\mathbb{Z}$) for n odd (resp. even).

The first step to define the general spline-based DQM was to consider the space V_n spanned by the translates $M_n(2 \cdot -j)$ in order to construct the minimally supported fundamental function L_n of the required interpolation operator $\mathcal{L}_n$. It has the form

$$L_n = \sum_{j \in J} c_j M_n(2 \cdot -j)$$

for some $c_j \in \mathbb{R}$, J being a finite subset of $\mathbb{Z}$. We impose that L_n satisfies the interpolation conditions

$$L(j) = \delta_{j,0}, \quad j \in \mathbb{Z}, \tag{1}$$

where δ stands for the Kronecker's delta.

Since the Laurent polynomials $\Phi_k(z) := \sum_{j \in \mathbb{Z}} M_n(2j+k) z^{2j}$, $k = 0, 1$, have no common zeros in $\mathbb{C} \setminus \{0\}$ (see [12,14]), it follows that (see [15]) any finite sequence c satisfying the identity

$$\Phi_0 \sum_{j \in \mathbb{Z}} c_{2j} e_{2j} + \Phi_1 \sum_{j \in \mathbb{Z}} c_{2j+1} e_{2j+1} = 1$$

provides such a function L_n. Here the notation $e_0(z) := 1$ and $e_k(z) := z^k$, $k \geq 1$ for the monomials is used. The following result on symmetric functions L_n with a small support was proved in [12] in a more general framework (see also [14]).

Proposition 1. *For each $n \geq 4$, let $J := \{-d_n, \ldots, d_n\}$ where*

$$d_n := \begin{cases} \lfloor r \rfloor - 2, & \text{for } n \text{ even,} \\ \lfloor r \rfloor - 1, & \text{for } n \text{ odd,} \end{cases}$$

and $\lfloor r \rfloor$ denotes the integer part of $r \in \mathbb{R}$. Then, there are coefficients a_j, $0 \leq j \leq 2d_n$ such that the function

$$L_n = a_0 M_n(2 \cdot +d_n) + \cdots + a_{2d_n} M_n(2 \cdot -d_n)$$

satisfies the interpolation conditions

$$L(j) = \delta_{j,0}, \quad j \in \mathbb{Z}.$$

It follows that

$$\operatorname{supp} L_n \subset \begin{cases} \left[-\frac{n}{2} + 1, \frac{n}{2} - 1\right], & \text{for } n \text{ even,} \\ \left[-\frac{n}{2} + \frac{3}{4}, \frac{n}{2} - \frac{3}{4}\right], & \text{for } n \text{ odd.} \end{cases}$$

Once defined the fundamental function L_n, the interpolation operator $\mathcal{L}_n$ is given by

$$\mathcal{L}_n(f) := \sum_{i \in \mathbb{Z}} f(i) L_n(\cdot - i).$$