Journal of Computational and Applied Mathematics 275 (2015) 356–365

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On inverses of infinite Hessenberg matrices

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ARTICLE INFO

Article history:
Received 9 July 2014
Received in revised form 16 July 2014

MSC:
15A09
15A99
47B36

Keywords:
Hessenberg matrix
Infinite matrix
Inverse matrix
Tridiagonal matrix

ABSTRACT

Here a known result on the structure of finite Hessenberg matrices is extended to infinite Hessenberg matrices. Its consequences for the example of infinite Hessenberg–Toeplitz matrices are described. The results are applied also to the inversion of infinite tridiagonal matrices via recurrence relations. Moreover, since there are available free parameters, different inverses can be associated with a given invertible tridiagonal matrix.

1. Introduction

A characterization of the finite nonsingular unreduced Hessenberg matrices is related with the distinguished rank structure of their inverse matrices [1–3]. Without loss of generality, we consider upper Hessenberg matrices. The inverses of such matrices are rank one perturbations, $\mathbf{UV}+\mathbf{T}$, of triangular matrices. The matrix $\mathbf{T}$ is (strictly) upper triangular and $\mathbf{U}$ and $\mathbf{V}$ are a column and a row vector, respectively. Their relevance to the case of finite tridiagonal matrices is immediate. We ask whether the inverses of real or complex infinite Hessenberg matrices have a similar structure, and we shall show that this is indeed the case. We propose a method for inverting infinite Hessenberg matrices that is based on these structural properties. In particular, classical inverses of general tridiagonal matrices can be generated through recurrence relations.

Some background about inversion of infinite matrices and their applications can be found in the literature; see e.g. [4–7] and the references given there. The classical role of the infinite unreduced Hessenberg matrices in orthogonal polynomials as matrix representation of the multiplication by z operator is well known [8]. In addition, infinite (transition or adjacency) Hessenberg matrices appear in signal processing, time series, and birth–death processes. Here the infinite Hessenberg matrices are regarded simply as matrices over $\mathbb{C}$ or $\mathbb{R}$. Nevertheless, the inversion of infinite Hessenberg matrices regarded as matrices representing bounded operators on the linear space ℓ^2 will be discussed briefly.

The outline is as follows. In Section 2, we recall some basic results about inverses of finite Hessenberg and tridiagonal matrices. In Section 3, we study the problem of the consistency of regarding Hessenberg matrices as the inverses of matrices

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http://dx.doi.org/10.1016/j.cam.2014.07.010

of the form $\mathbf{B} = \mathbf{UV} + \mathbf{T}$. In Section 4, we propose a method for inverting infinite unreduced Hessenberg matrices, in particular Hessenberg–Toeplitz matrices and tridiagonal matrices. Its application to infinite reduced Hessenberg matrices with finitely many zeros on their subdiagonals is straightforward. In addition, recurrence relations for evaluating classical inverses of tridiagonal matrices are introduced. Finally, Section 5 contains a short remark regarding the inversion of bounded linear operators. Throughout the text the results are illustrated with appropriate examples.

2. Inverses of finite Hessenberg matrices

2.1. Unreduced Hessenberg matrices of finite order

A matrix $\mathbf{H}$ is upper Hessenberg if its elements h_{ij} satisfy $h_{ij} = 0$ for $i \geq j + 2$. Here we extend and adapt a well-known lemma [1–3] to upper Hessenberg matrices. We also recall that an order n Hessenberg matrix $\mathbf{H} = (h_{ij})_{i,j=1}^{n}$ is an unreduced upper Hessenberg matrix if it has nonzero subdiagonal entries, $h_{i+1,i} \neq 0,\ i = 1, 2, \ldots, n-1$.

Lemma 1. *An $n \times n$ nonsingular matrix $\mathbf{H} = (h_{ij})_{i,j=1}^{n}$ is unreduced upper Hessenberg if and only if its inverse matrix has the form $\mathbf{B} = \mathbf{UV} + \mathbf{T}$, where $\mathbf{U}$ is a column matrix with nonzero nth component and $\mathbf{V}$ is a row matrix with nonzero first component. The matrix $\mathbf{T}$ is strictly upper triangular, having zero entries on its main diagonal and nonzero entries $t_{i,i+1} = h_{i+1,i}^{-1} \neq 0$, $1 \leq i \leq n-1$, on the superdiagonal.*

See [2] for a proof. The matrix $\mathbf{B}$ has the form

$$\mathbf{B} = \begin{pmatrix} u_1 v_1 & b_{12} & b_{13} & \cdots & b_{1n} \\ u_2 v_1 & u_2 v_2 & b_{23} & \cdots & b_{2n} \\ u_3 v_1 & u_3 v_2 & u_3 v_3 & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_n v_1 & u_n v_2 & u_n v_3 & \cdots & u_n v_n \end{pmatrix},$$

where, for $j > i$, $b_{ij} = u_i v_j + t_{ij}$. The determinant $|\mathbf{B}|$ is given by

$$|\mathbf{B}| = \frac{v_1 u_n}{\prod_{i=1}^{n-1} (-h_{i+1,i})} = (-1)^{n-1} v_1 u_n \prod_{i=1}^{n-1} t_{i,i+1}.$$

Since $\mathbf{B}$ is nonsingular, the triangular matrix $\mathbf{T}$ must have nonzero entries on its superdiagonal. The components of the vectors $\mathbf{U}$ and $\mathbf{V}$ are

$$u_i = \frac{(-1)^{i-1}}{|\mathbf{H}|} |\mathbf{H}_{n-i}^{(i)}| \frac{1}{\prod_{k=i}^{n-1} h_{k+1,k}}, \qquad v_j = (-1)^{j-1} |\mathbf{H}_{j-1}| \prod_{k=j}^{n-1} h_{k+1,k}.$$

A formally equivalent lemma holds for lower Hessenberg matrices. The entry b_{ij} of $\mathbf{B}$ has the determinantal representation

$$b_{ij} = \begin{cases} (-1)^{i+j} \dfrac{|\mathbf{H}_{j-1}| \cdot |\mathbf{H}_{n-i}^{(i)}| \cdot [h_{i,i-1} \cdots h_{j+1,j}]}{|\mathbf{H}|}, & \text{if } i \geq j; \\ (-1)^{i+j} \dfrac{|\mathbf{H}_{j-1}| \cdot |\mathbf{H}_{n-1}^{(i)}|}{|\mathbf{H}| \cdot [h_{j,j-1} \cdots h_{i+1,i}]} - \dfrac{(-1)^{i+j} |\mathbf{H}_{j-i-1}^{(i)}|}{[h_{j,j-1} \cdots h_{i+1,i}]}, & \text{if } i < j, \end{cases} \tag{1}$$

where $|\mathbf{H}_{j-1}|$ is the $(j-1)$st left principal minor, and $|\mathbf{H}^{(i)}|_{n-i}$ and $|\mathbf{H}^{(i)}|_{j-i-1}$ are the right principal minors of the matrices $\mathbf{H}_n$ and $\mathbf{H}_{j-i-1}$, respectively; see Corollary 1 in [9]. It follows immediately that, for $i < j$, the entries t_{ij} of $\mathbf{T}$ have the form

$$t_{ij} = \frac{(-1)^{i+j+1} |\mathbf{H}_{j-i-1}^{(i)}|}{\prod_{k=j-1}^{i} h_{k+1,k}}. \tag{2}$$

2.2. Unreduced tridiagonal matrices of finite order

Recall that a tridiagonal matrix having nonzero entries on both the subdiagonal and the superdiagonal is called an unreduced tridiagonal matrix. The following result is also well known [1–3,10].