Journal of Computational and Applied Mathematics 275 (2015) 366–374

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

# One-point Newton-type iterative methods: A unified point of view[☆]

Alicia Cordero[*], Cristina Jordán, Juan R. Torregrosa

*Instituto de Matemáticas Multidisciplinar, Universitat Politècnica de València, Camino de Vera, s/n, 46022 València, Spain*

ARTICLE INFO

*Article history:*
Received 2 July 2014
Received in revised form 14 July 2014

*MSC:*
65H05

*Keywords:*
Nonlinear equation
One-point method
Newton's method
Order of convergence
Efficiency index
Optimal order

ABSTRACT

In this paper, a unified point of view that includes the most of one-point Newton-type iterative methods for solving nonlinear equations is introduced. A simple idea to design iterative methods with quadratic or cubic convergence is also described. This idea is extended to construct one-point iterative methods of order four. In addition, several numerical examples are given to illustrate and compare different known methods and some introduced by using this unifying idea.



## 1. Introduction

Solving nonlinear equations is a classical problem which has interesting applications in several branches of science and engineering. Many optimization problems such as searching for a local minimizer of function [1], the potential equations in the transonic regime of dense gases in gas dynamics [2] and the boundary value problems encountered in kinetic theory of gases [3], elasticity [4] and problems in other applied areas can be reduced to nonlinear equations. In general, to compute their roots we must drawn on to iterative methods.

This paper is concerned with iterative methods to find a simple root $\alpha$ of a nonlinear equation $f(x) = 0$, where $f$ is a real function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$, defined in an open interval $I$. There are many iterative methods such as Newton's method, Halley and super-Halley's schemes, Chebyshev's method, and their variants (see [5] and the references therein). In the following, some basics concepts are introduced, that can be found in [1,5]. Newton's method is the best known and probably the most used algorithm for solving $f(x) = 0$. It is given by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \quad k = 0, 1, \ldots$$

which converges quadratically in some neighborhood of $\alpha$, that is, there exists a positive constant $C$ such that $\lim_{k\to\infty} \frac{|x_{k+1}-\alpha|}{|x_k-\alpha|^2} = C$. More generally, for the sequence $\{x_k\}_{k=0}^{\infty}$ generated by an iterative method, if there exist positive constants $C$ and $p$ such that

☆ This research was supported by Ministerio de Ciencia y Tecnología MTM2011-28636-C02-02 and Universitat Politècnica de València SP20120474.

* Corresponding author. Tel.: +34 9638 77190; fax: +34 9638 77199.
*E-mail addresses:* acordero@mat.upv.es (A. Cordero), cjordan@mat.upv.es (C. Jordán), jrtorre@mat.upv.es (J.R. Torregrosa).

http://dx.doi.org/10.1016/j.cam.2014.07.009
0377-0427/

$$\lim_{k\to\infty} \frac{|x_{k+1} - \alpha|}{|x_k - \alpha|^p} = C,$$

then the method is said to converge to $\alpha$ with the local order of convergence $p$.

Commonly, the efficiency of an iterative method is measured by the efficiency index, defined by Ostrowski [6] as $p^{1/d}$, where $p$ is the order of convergence and $d$ is the number of functional evaluations per iteration. Kung and Traub conjectured in [7] that the order of convergence of any iterative method, without memory, cannot exceed the bound $2^{d-1}$, called the optimal order. Let us recall that an iterative method without memory is a scheme whose $(k + 1)$th iteration is obtained by using only the previous $k$th iteration. The efficiency index of Newton's method is 1.414, as it uses two functional evaluations (one of $f$ and another one of $f'$) and its order of convergence is two. So, it is an optimal scheme.

The construction of numerical methods for solving nonlinear equations is an interesting task, which has attracted the attention of many authors for more than three centuries. These schemes can be classified in two big families: one-point and multipoint schemes, depending on if $(k + 1)$th iteration is obtained by using functional evaluations only of iteration $k$ or also functional evaluations in other intermediate points, respectively. The classical methods mentioned before are one-point schemes. During the later years, numerous papers devoted to design one-point iterative methods for solving nonlinear equations, $f(x) = 0$, have appeared. These methods are developed from the classical algorithms by using Taylor interpolating polynomials, quadrature rules or some other techniques. However, due to the limitations and restrictions of the one-point iterative schemes, multipoint methods appeared in the literature. A good survey of these multipoint schemes can be found in [8].

In this work, a systematic treatment of the one-point iterative methods by using the weight function procedure is provided. We can include, under this unified point of view, all the one-point known methods, as far as we know, of orders two and three.

The rest of the paper is organized as follows: in Section 2, many known one-point methods or classes of schemes of order two are shown, giving a general expression of them by using a real weight function $H$, that depends on a particular variable, and we analyze the conditions that function $H$ must satisfy in order to obtain iterative methods of order two. Section 3 is devoted to the same idea as Section 2 but for iterative schemes of order three. In Section 4, the procedure carried out in the previous section for increasing the order of convergence is generalized. Several numerical examples are given in Section 5 to illustrate and compare the efficiency of the methods considered in the paper.

## 2. One-point methods of second order

In this section we are going to show some of the classical and more recent methods or families of schemes for finding a root of the equation $f(x) = 0$, with second order of convergence. Their iterative expressions will be described as modified Newton's method with a weight function $H$, depending on variable $u(x) = \frac{f(x)}{f'(x)}$.

Kanwar and Tomar proposed in [9] the following parametric family of second order iterative methods

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k) + \beta f(x_k)} = x_k - \frac{f'(x_k)}{f'(x_k) + \beta f(x_k)} \frac{f(x_k)}{f'(x_k)} = x_k - \frac{1}{1 + \beta u(x_k)} \frac{f(x_k)}{f'(x_k)}, \tag{1}$$

where $\beta$ is a parameter, derived by expanding a particular function in Taylor series. Let us observe that for $\beta = 0$ we obtain Newton's scheme.

From this idea, Kou and Li in [10] described the bi-parametric family of methods of order two

$$x_{k+1} = x_k - \left(1 + \frac{\lambda h(x_k)}{1 + \beta h(x_k)}\right) h(x_k),$$

where $\lambda, \beta$ are parameters and $h(x_k) = \frac{f(x_k)}{f'(x_k)+\beta f(x_k)}$. We can transform this iterative expression in

$$x_{k+1} = x_k - \left(1 + \frac{\lambda u(x_k)}{(1 + \beta u(x_k))(1 + 2\beta u(x_k))}\right) \frac{f(x_k)}{f'(x_k)}. \tag{2}$$

In [11], Noor presented the following method with quadratic convergence:

$$\begin{aligned} x_{k+1} &= x_k - \frac{2f(x_k)}{f'(x_k) + \sqrt{f'(x_k)^2 + 4\beta^3 f(x_k)^3}} = x_k - \frac{2}{1 + \sqrt{1 + 4\beta^3 f(x_k)^3/f'(x_k)^2}} \frac{f(x_k)}{f'(x_k)} \\ &= x_k - \frac{2}{1 + \sqrt{1 + 4\beta^3 f(x_k) u(x_k)^2}} \frac{f(x_k)}{f'(x_k)} \end{aligned} \tag{3}$$

obtained by using Taylor polynomials along with an auxiliary equation.

All these methods are optimal in the sense of Kung–Traub's conjecture, since they have order two and use two functional evaluations per step. So, their efficiency index is the same, but not the number of floating point operations, which depends