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Error analysis in the reconstruction of a convolution kernel in a semilinear parabolic problem with integral overdetermination

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ABSTRACT

A semilinear parabolic problem of second order with an unknown solely time-dependent convolution kernel is considered. An additional given global measurement (a space integral of the solution) ensures the existence of a unique weak solution. The unknown kernel function can be approximated by a time-discrete numerical scheme based on Backward Euler's method (Rothe's method). In this contribution, an error analysis for the time discretization is performed of the existing numerical algorithm. Numerical experiments support the theoretically obtained results.

1. Introduction

In this contribution, the domain Ω is a bounded Lipschitz domain in $\mathbb{R}^N, N \geq 1$, with $\partial\Omega = \Gamma$ and $\Theta = [0, T], T > 0$, the time frame. The aim of this paper is to derive estimates for the distance between the discrete and continuous solution of a semilinear parabolic problem. The former is based on a time-discrete numerical scheme, described in [1], that approximates the solution of the following semilinear parabolic problem: determine the solution u and the convolution kernel $K(t)$ such that

$$\begin{cases} \partial_t u(\mathbf{x}, t) - \Delta u(\mathbf{x}, t) + K(t)h(\mathbf{x}, t) + (K * u(\mathbf{x}))(t) = f(u(\mathbf{x}, t), \nabla u(\mathbf{x}, t)), & \text{in } \Omega \times \Theta, \\ -\nabla u(\mathbf{x}, t) \cdot \boldsymbol{\nu} = g(\mathbf{x}, t), & \text{on } \Gamma \times \Theta, \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), & \text{in } \Omega \end{cases} \tag{1}$$

when an additional global measurement

$$\int_\Omega u(\mathbf{x}, t)\mathrm{d}\mathbf{x} = m(t) \tag{2}$$

is satisfied. Note that the data functions $h : \Omega \times \Theta \to \mathbb{R}, f : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}, g : \Gamma \times \Theta \to \mathbb{R}, u_0 : \Omega \to \mathbb{R}$ and $m : \Theta \to \mathbb{R}$ are known, and time-convolution is defined as

$$(K * u(\mathbf{x}))(t) = \int_0^t K(t-s)\, u(\mathbf{x}, s)\mathrm{d}s, \quad t \in \Theta.$$

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Regarding f, one can replace it with $f + \varphi$ where $\varphi : \Omega \times \Theta \to \mathbb{R}$ is sufficiently regular. Such type of problems arise in the theory of reactive contaminant transport. In [2] one considers the following differential equation

$$\partial_t C + \nabla \cdot (\mathbf{V}C) - \Delta C = \frac{-\rho_b}{n}\partial_t S \tag{3}$$

for the aqueous concentration C and sorbed concentration per unit mass of solid S with mass transformation rate in first order form of

$$\partial_t S = K_r(K_d C - S)$$

with desorption rate K_r and equilibrium distribution coefficient K_d. This can be formally solved as

$$S(t) = \mathrm{e}^{-K_r t}S(0) + K_r K_d \int_0^t \mathrm{e}^{-K_r(t-\xi)}C(\xi)\mathrm{d}\xi.$$

Therefore, (3) can be rewritten as problem (1) for $u = C$ with $K(t) = -\frac{\rho_b}{n}K_r^2 K_d \mathrm{e}^{-K_r t}$ and $h(t) = \frac{S(0)}{K_r K_d}$. For an overview in the literature of papers dealing with integral overdetermination one may refer to [3–11]. Denote by $(\cdot,\cdot)$ the standard inner product of $\mathrm{L}^2(\Omega)$ and $\|\cdot\|$ its induced norm. The variational formulation of problem (1) reads as:

find $\langle u(t), K(t)\rangle \in \mathrm{H}^1(\Omega) \times \mathbb{R}$ with $\partial_t u(t) \in \mathrm{L}^2(\Omega)$ such that for all $\phi \in \mathrm{H}^1(\Omega, \mathbb{R})$ it holds

$$(\partial_t u, \phi) + (\nabla u, \nabla \phi) + (g, \phi)_\Gamma + K(t)(h, \phi) + (K * u, \phi) = (f(u, \nabla u), \phi), \quad \text{a.e. } t \in \Theta, \tag{P}$$

and such that the global measurement (2) is satisfied.

If we set $\phi = 1$ in (P) we obtain together with $(u, 1) = m(t)$

$$m'(t) + (g, 1)_\Gamma + K(t)(h, 1) + K * m = (f(u, \nabla u), 1). \tag{MP}$$

In [1], the authors proved the following existence and uniqueness theorem for the inverse problem:

Theorem 1 (*See [1]*). *Suppose f is bounded and Lipschitz continuous in all variables, $g \in \mathrm{C}^1(\Theta, \mathrm{L}^2(\Gamma))$, $h \in \mathrm{C}^0(\Theta, \mathrm{H}^1(\Omega)) \cap \mathrm{C}^1(\Theta, \mathrm{L}^2(\Omega))$ and $\min_{t\in\Theta} |(h(t), 1)| \geq \omega > 0$, $m \in \mathrm{C}^2(\Theta, \mathbb{R})$ and $u_0 \in \mathrm{H}^2(\Omega)$. Then there exists a unique couple solutions $\langle u, K\rangle$ to (P)–(MP), where $u \in \mathrm{C}(\Theta, \mathrm{H}^1(\Omega))$, $\partial_t u \in \mathrm{L}^\infty(\Theta, \mathrm{L}^2(\Omega))$ and $K \in \mathrm{C}(\Theta)$, $K' \in \mathrm{L}^2(\Theta)$.*

The outline of this paper is as follows. In Section 2, a time-discrete scheme to approximate the solution to problem (1)–(2) is described. The corresponding error estimates are derived in Section 3. Finally, some numerical experiments are developed in Section 4.

2. Numerical scheme

2.1. Discretization

We apply the Rothe method [12,13]. Consider an equidistant time-partitioning of the time frame Θ with a step $\tau = T/n < 1$, for any $n \in \mathbb{N}$. We use the notation $t_i = i\tau$ and for any function z we write

$$z_i = z(t_i), \qquad \delta z_i = \frac{z_i - z_{i-1}}{\tau}.$$

At time t_i we infer from (P) the backward Euler scheme

$$(\delta u_i, \phi) + (\nabla u_i, \nabla \phi) + (g_i, \phi)_\Gamma + K_i(h_i, \phi) + \sum_{k=1}^{i}(K_k u_{i-k}\tau, \phi) = (f_{i-1}, \phi) \tag{DPi}$$

where $f_i = f(u_i, \nabla u_i)$. This is conveniently written as $B(u_i, \phi) = F_i(\phi)$ with

$$B(u_i, \phi) = \frac{1}{\tau}(u_i, \phi) + (\nabla u_i, \nabla \phi), \quad F_i(\phi) = (f_{i-1}, \phi) - (g_i, \phi)_\Gamma - K_i(h_i, \phi) - \sum_{k=1}^{i}(K_k u_{i-k}\tau, \phi) + \frac{1}{\tau}(u_{i-1}, \phi).$$

Analogously, we obtain from (MP)

$$m_i' + (g_i, 1)_\Gamma + K_i(h_i, 1) + \sum_{k=1}^{i} K_k m_{i-k}\tau = (f_{i-1}, 1). \tag{DMPi}$$

Using (DPi) and (DMPi) the numerical algorithm is as follows: