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Convergence, efficiency and dynamics of new fourth and sixth order families of iterative methods for nonlinear systems

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ABSTRACT

In this work we present a new family of iterative methods for solving nonlinear systems that are optimal in the sense of Kung and Traub's conjecture for the unidimensional case. We generalize this family by performing a new step in the iterative method, getting a new family with order of convergence six. We study the efficiency of these families for the multidimensional case by introducing a new term in the computational cost defined by Grau-Sánchez et al. A comparison with already known methods is done by studying the dynamics of these methods in an example system.

1. Introduction

Finding iterative methods with high order of convergence in order to approximate the solution of a nonlinear system $F(x) = 0$ is an active field in numerical analysis. Nowadays, the range of applications where it is required to use a high level of numerical precision is increasing.

In the scalar case, a recent publication, [1], makes an interesting compilation of multipoint iterative methods and analyzes their efficiency, accuracy and optimality.

Focusing on higher order iterative methods for the multidimensional case, we can mention, among others, some recently published works [2–7], where, as we can see, different techniques can be applied in order to improve the computational cost and so the effectiveness of the procedures for approximating solutions of nonlinear systems.

In this work we generalize the technique used in [8], obtaining a new family of iterative methods with fourth order of convergence. The procedures used in [9,6] for increasing the convergence order of an iterative method, that is, to perform another Newton's step avoiding the evaluation of the Jacobian matrix in order to get the maximum efficiency, do not work for the optimal method introduced in [8], so we propose a new procedure to increase the order with a reasonable efficiency.

Obviously, performing a new step in an iterative method carries more function evaluations and so one has to check if the gain in convergence order justifies the increase of the computational cost. A thorough study of the cost and efficiency of iterative methods for nonlinear systems can be found in [5,10]. Nevertheless, we introduce a new term in the cost expression to take into account matrix–vector operations that occur in some iterative methods such as considered here.

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The paper is organized as follows. New families of iterative methods, Jarratt's two point and three point methods, are obtained in Section 2. In Section 3, we analyze the computational efficiency for the new methods and in Section 4 the new methods are applied in order to approximate the solutions of some nonlinear systems. Finally, Section 5 studies the dynamics of these methods for a particular nonlinear system and Section 6 is devoted to the conclusions.

2. New families of iterative methods

Our aim is to develop high order methods for nonlinear systems, motivated by the techniques exposed in Section 2.6 of Chapter 3 of [1] for obtaining multipoint iterative methods of Jarratt's type in the unidimensional case. We try to apply some of the ideas of [11,12] to the fourth order method recently published by Sharma et al. [8]. First of all, we generalize this technique by introducing a new term in their proposal, obtaining a new family of fourth order iterative methods.

That is, we consider the family of methods given by:

$$y_n = x_n - \theta \Gamma_{x_n} F(x_n) \tag{1}$$

$$H(x_n, y_n) = \Gamma_{x_n} F'(y_n) \tag{2}$$

$$G_s(x_n, y_n) = s_1 I + s_2 H(y_n, x_n) + s_3 H(x_n, y_n) + s_4 H(y_n, x_n)^2 \tag{3}$$

$$z_n = x_n - G_s(x_n, y_n) \Gamma_{x_n} F(x_n) \tag{4}$$

$$x_{n+1} = z_n \tag{5}$$

where $\Gamma_{x_n} = F'(x_n)^{-1}$, and $\theta, s_1, s_2, s_3, s_4$ are constants that we determine in order to get a new family of fourth order optimal methods. Notice that, in the unidimensional case, we evaluate just three functions, $F(x_n)$, $F'(x_n)$ and $F'(y_n)$, so that the family is optimal in the sense of Kung and Traub's conjecture [13].

By adequately using Taylor's expansion we prove the following result about the convergence order.

Theorem 1. *Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a sufficiently Fréchet differentiable function in a convex neighborhood of D, containing α, that is a solution of the system $F(x) = 0$, whose Jacobian matrix is continuous and nonsingular in D. Then, for an initial approximation sufficiently close to α, the family of methods defined by (1)–(5) has local order of convergence 4 for the following relations among the parameters: $s_1 = \frac{5-8s_2}{8}$, $s_3 = \frac{s_2}{3}$, $s_4 = \frac{9-8s_2}{24}$; $\forall s_2 \in \mathbb{R}$ and for $\theta = \frac{2}{3}$.*

The error equation obtained is as follows:

$$e_{n+1} = \frac{(64s_2 + 117)c_2^3 - 81c_1c_3c_2 + 9c_1^2c_4}{81c_1^3}\, e_n^4 + O\left(e_n^5\right)$$

where $e_n = x_n - \alpha$ and $c_k = \frac{F^{(k)}(\alpha)}{k!}$, $k \geq 1$.

Proof. By applying Taylor's expansion of $F(x_n)$ about α and taking into account that $F(\alpha) = 0$, we have

$$F(x_n) = c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + O(e_n^5) \tag{6}$$

where $c_k = \frac{F^{(k)}(\alpha)}{k!} \in \mathcal{L}_k(\mathbb{R}^n, \mathbb{R}^n)$, $k \geq 1$. By differentiating, one has

$$F'(x_n) = c_1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + O(e_n^5) \tag{7}$$

and then the following Taylor's development:

$$\Gamma_{x_n} F(x_n) = e_n - \frac{c_2}{c_1}\, e_n^2 + \frac{2\left(c_2^2 - c_1c_3\right)}{c_1^2}\, e_n^3 + \frac{-4c_2^3 + 7c_1c_2c_3 - 3c_1^2c_4}{c_1^3}\, e_n^4 + O(e_n^5).$$

By substituting in the first step (1), we have:

$$y_n - \alpha = (1-\theta)\, e_n + \frac{c_2\, \theta}{c_1}\, e_n^2 + \frac{2\,\theta\left(c_1c_3 - c_2^2\right)}{c_1^2}\, e_n^3 + \frac{\theta\left(4c_2^3 - 7c_1c_2c_3 + 3c_1^2c_4\right)}{c_1^3}\, e_n^4 + O(e_n^5).$$

And, then

$$\begin{aligned} F'(y_n) = {} & c_1 - 2\left(c_2(\theta-1)\right) e_n + \left(3c_3(\theta-1)^2 + \frac{2c_2^2\theta}{c_1}\right) e_n^2 - \frac{2}{c_1^2}\Big(2c_1^2c_4(\theta-1)^3 \\ & + 2c_2^3\theta + c_1c_2c_3\theta(3\theta-5)\Big)e_n^3 + \frac{1}{c_1^3}\Big(6c_1^2c_4c_2\theta\left(2\theta^2 - 4\theta + 3\right) + 8c_2^4\theta \\ & + c_1c_3c_2^2\theta(15\theta-26) + c_1^2(\theta-1)\left(5c_1c_5(\theta-1)^3 - 12c_3^2\theta\right)\Big)e_n^4 + O(e_n^5) \end{aligned} \tag{8}$$