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A nonlinear parabolic integro-differential problem with an unknown Dirichlet boundary condition

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ABSTRACT

A nonlinear parabolic integro-differential equation $\partial_t g(u) - \Delta u = F + \int_0^t f(s, u(s))\, \mathrm{d}s$ with a known Neumann boundary condition on a part of the boundary and an unknown Dirichlet boundary condition $\alpha(t)$ on the other part of the boundary is studied. The inverse problem of identifying the unknown time-dependent function $\alpha(t)$ from an additional integral measurement $E(t) = \int_\Omega g(u(t, \boldsymbol{x}))\, \mathrm{d}\boldsymbol{x}$ is investigated. The well-posedness of the problem in suitable function spaces is shown and a numerical time-discrete scheme for approximations is designed. Convergence of the proposed scheme is supported by a numerical experiment.

1. Introduction

In the last decades, the field of inverse problems (IPs) has undergone rapid development. The enormous technological progress together with the development of new efficient numerical methods made it possible to simulate real-world direct problems of growing complexity. Many applications in science and engineering lead to IPs, which in turn stimulated mathematical research, e.g., on uniqueness questions and on developing stable and powerful numerical methods for solving them. In literature, three main types of inverse problems are distinguished: (a) parameter identification, where the material parameters appearing in the equation are not known and should be reconstructed, e.g., diffusion coefficients, source terms, etc., (b) boundary value inverse problems, where direct measurements on the boundary (or a part of it) are unfeasible and have to be determined, (c) evolutionary inverse problems in which the initial conditions are not known and have to be reconstructed.

This work is devoted to the reconstruction of missing Dirichlet data on a part of the boundary from an additional integral measurement. Let $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be a bounded domain with a Lipschitz continuous boundary Γ that is split into two non-overlapping parts Γ_1 and Γ_2 with $|\Gamma_2| > 0$. By $\boldsymbol{\nu}$ we denote the outward unit normal vector on Γ. The governing problem is represented by a nonlinear parabolic differential equation with a memory term along with initial data and mixed boundary conditions (BCs), more exactly

$$\begin{cases} \partial_t g(u) - \Delta u = F + \int_0^t f(s, u(s))\, \mathrm{d}s & \text{in } (0, T) \times \Omega; \\ -\nabla u \cdot \boldsymbol{\nu} = h & \text{on } (0, T) \times \Gamma_1; \\ u = \alpha & \text{on } (0, T) \times \Gamma_2; \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \tag{1}$$

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If all data functions appearing in (1) are given and fulfill appropriate conditions, then there exists a unique solution to (1). This can be proved by means of standard techniques in suitable function spaces. In this contribution, we study a situation in which the coefficient α is unknown and is supposed to be a function of time only, i.e. $\alpha = \alpha(t)$. Note that in one-dimension and for $\Gamma_1 = \emptyset$, this Dirichlet boundary condition is equivalent to the periodic boundary condition $u(t, 0) = u(t, 1)$ if $\Omega = (0, 1)$. Our goal is to find $\{u, \alpha\}$ from the additional integral measurement

$$\int_\Omega g(u(t, \boldsymbol{x}))\, \mathrm{d}\boldsymbol{x} = E(t) \quad \text{in } (0, T). \tag{2}$$

Parabolic integro-differential equations have some applications in reactive contaminant transport in the saturated zone, cf. [1, Chap. 15]. The governing PDE is nothing else than the continuity equation. Adopting the Darcy law for diffusion, one can get a parabolic PDE for the contaminant concentration $u(t)$. Reaction transformation can then be formally captured via augmenting the source by a generalized mass loss rate $\partial_t s$, that depends on the absorbed concentration s. The usual linear first-order form is

$$\partial_t s = K_r(K_d u - s),$$

with some given constants K_r and K_d. This can be formally resolved as

$$s(t) = e^{-K_r t} s(0) + K_r K_d \int_0^t e^{-K_r(t-\xi)} u(\xi)\, \mathrm{d}\xi.$$

We can see that the right-hand side of the governing PDE for u will depend on the time integral of u. The Dirichlet BC on Γ_2 then models contact with another medium with fast diffusion. Therefore, the value α is a space-constant along Γ_2, but it can vary in time. Furthermore, the function $E(t)$ represents the total mass of contaminant inside Ω. For a more comprehensive overview of literature dealing with integral overdetermination within the framework of IPs for parabolic, hyperbolic and Navier–Stokes equations, we refer the reader e.g. to [2–8] and the references therein.

From now on, we assume that f is a global Lipschitz continuous function in all variables. Moreover, the function g is assumed to be a monotonically increasing continuous function obeying

$$g(0) = 0, \qquad 0 < \delta \leqslant g'(s), \qquad |g(s)| \leqslant C\,(1 + |s|)\,, \quad \forall s \in \mathbb{R}. \tag{3}$$

The added value of this paper relies on the global (in time) solvability of the IP along with the nonlinear character of the governing partial differential equation and on the designed numerical scheme for approximations. In Section 2, we study the uniqueness of a solution to (1)–(2) in appropriate function spaces. In Section 3, we design an time-discrete approximation scheme based on Rothe's method, cf. [9]. Section 4 is devoted to the existence of a solution to system (1)–(2) and to the convergence of the approximations towards the unique weak solution. A numerical experiment is described in Section 5 to support the theoretically obtained results. Finally, a conclusion is stated in Section 6.

Remark. The values C, ε and C_ε are considered generic and positive constants (independent of the discretization parameter), where ε is arbitrarily small and $C_\varepsilon = C\left(\frac{1}{\varepsilon}\right)$ arbitrarily large. We will use the same notation for different constants, but the meaning will be clear from the context.

2. Variational formulation and uniqueness

We shall work in a variational framework. We denote by (u, v) the usual L_2-inner product of real-valued functions u and v in Ω, i.e. $(u, v) = \int_\Omega u(\boldsymbol{x})v(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}$, $\boldsymbol{x} \in \Omega$, and $\|v\| = \sqrt{(v, v)}$. The L_2-inner product on the boundary Γ will be written as $(u, v)_\Gamma = \int_\Gamma u(\boldsymbol{x})v(\boldsymbol{x})\, \mathrm{d}\Gamma$, $\boldsymbol{x} \in \Gamma$.

According to the assumption that the Dirichlet trace of a solution is a space-constant on Γ_2, we will seek the solution in the space $V := \left\{\varphi \in \mathrm{H}^1(\Omega)\,;\ \varphi_{|\Gamma_2} \text{ is constant}\right\}$, i.e. $u(t) \in V$. This is a closed subspace of the standard Hilbert space $\mathrm{H}^1(\Omega)$ with the inherited norm from $\mathrm{H}^1(\Omega)$, i.e. $\|u\|_1^2 = \|u\|^2 + \|\nabla u\|^2$, cf. [10]. We will write V^* for the dual space of V.

Multiplying the differential equation of (1) with a test function $\varphi \in V$, integrating the result over Ω and applying the Green theorem gives us

$$(\partial_t g(u), \varphi) + (\nabla u, \nabla \varphi) = (F, \varphi) - (h, \varphi)_{\Gamma_1} + \varphi_{|\Gamma_2}\, (\nabla u \cdot \boldsymbol{\nu}, 1)_{\Gamma_2} + \left(\int_0^t f(s, u(s))\, \mathrm{d}s, \varphi\right), \quad \forall \varphi \in V. \tag{4}$$

The crucial trick is to get rid of the total flux through Γ_2, i.e. $(\nabla u \cdot \boldsymbol{\nu}, 1)_{\Gamma_2}$. This means constructing a variational problem with u as the only unknown. We get an expression for $(\nabla u \cdot \boldsymbol{\nu}, 1)_{\Gamma_2}$ by setting $\varphi = 1$ in (4) and using the measurement (2). More explicitly,

$$(\nabla u \cdot \boldsymbol{\nu}, 1)_{\Gamma_2} = E' - (F, 1) + (h, 1)_{\Gamma_1} - \left(\int_0^t f(s, u(s))\, \mathrm{d}s, 1\right).$$